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CONSTRUCTION OF p-WAVELET PACKETS

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Abstract: In this paper, we introduce nonuniform multiresolution p-analysis on $L^2(\mathbb{R}^+)$, and then we construct the associated p-wavelet packets for such a multiresolution p-analysis.

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1. Introduction

Recently, Gabardo and Nashed [2] defined the concept of a multiresolution analysis (MRA) where the associated translation set is a discrete set which is not necessarily a group. More precisely, this set is of the form $\{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N - 1$, r is an odd integer relatively prime to N. They call this a nonuniform MRA. In this article, we construct the nonuniform p-wavelet packets associated with nonuniform multiresolution p-analysis.

All the definitions and properties in this section can also be found in [1, 3]. For x in $\mathbb{R}^+$ and any positive integer j, we set

$$x_j = [p^j x] \pmod{p}, \quad x_{-j} = [p^{1-j} x] \pmod{p}. \tag{1.1}$$

Consider on $\mathbb{R}^+$ the addition defined as follows:

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$$x \oplus y = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j}$$

with

$$\zeta_j = x_j + y_j \quad (\mathrm{mod}\ p) \quad (j \in \mathbb{Z}\backslash\{0\}),$$

where $\zeta_j \in \{0, 1, \cdots, p-1\}$ and x_j, y_j are calculated by (1.1). Moreover, we note that $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes the subtraction modulo p in $\mathbb{R}^+$.

For $x \in [0, 1)$, let

$$r_0(x) = \begin{cases} 1, & x \in [0, 1/p), \\ \varepsilon_p^l, & x \in [lp^{-1}, (l+1)p^{-1}) \quad (l = 1, \cdots, p-1), \end{cases}$$

where $\varepsilon_p = \exp(2\pi i/p)$. The extension of the function r_0 to $\mathbb{R}^+$ is denoted by the equality $r_0(x+1) = r_0(x)$, $x \in \mathbb{R}^+$. Then the generalized Walsh functions $\{w_m(x) : m \in \mathbb{Z}\}$ are defined by

$$w_0(x) \equiv 1, \qquad w_m(x) = \prod_{j=0}^{k} (r_0(p^j x))^{\mu_j},$$

where

$$m = \sum_{j=0}^{k} \mu_j p^j, \qquad \mu_j \in \{0, 1, ..., p-1\}, \ \ \mu_k \neq 0.$$

For $x, w \in \mathbb{R}^+$, let

$$\chi(x, w) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j w_{-j} + x_{-j} w_j)\right),$$

where x_j, w_j are given by (1.1). Note that $\chi(x, m/p^{n-1}) = \chi(x/p^{n-1}, m) = w_m(x/p^{n-1})$ for all $x \in [0, p^{n-1})$, $m \in \mathbb{Z}^+$.

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x) \overline{\chi(x, \xi)} dx,$$

Definition 1.1. Let N be an integer, $N \geq 1$, and $\Lambda = \{0, r/N\} + p\mathbb{Z}$, where r is an odd integer relatively prime to N with $1 \leq r \leq pN-1$. A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^+)$ will be called a nonuniform multiresolution p-analysis associated with Λ if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$, for all j $\in \mathbb{Z}$;

(ii) $f \in V_j$ if and only if $f(pN\cdot) \in V_{j+1}$;

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iv) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^+)$;

(v) There exists a function $\varphi \in V_0$, called the scaling function, such that $\{\varphi(x \ominus \lambda) : \lambda \in \Lambda,\ x \in R^+\}$ forms an orthonormal basis for V_0.

Let $W_j = V_{j+1} \ominus V_j$, $j \in \mathbb{Z}$. These subspaces inherit the scaling property of V_j, namely

$$f \in W_j \quad \text{if and only if} \quad f(pN\cdot) \in W_{j+1}. \tag{1.2}$$

Moreover, the subspaces W_j are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{R}^+) = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \oplus (\bigoplus_{j\geq 0} W_j). \tag{1.3}$$

A set of functions $\{\psi_1, \psi_2, ..., \psi_{pN-1}\}$ in $L^2(\mathbb{R}^+)$ is said to be a set of basic p-wavelets associated with the nonuniform multiresolution p-analysis if the collection $\{\psi_l(x \ominus \lambda) : 1 \leq l \leq pN-1,\ \lambda \in \Lambda,\ x \in R^+\}$ forms an orthonormal basis for W_0.

In view of (1.2) and (1.3), it is clear that if $\{\psi_1, \psi_2, ..., \psi_{pN-1}\}$ is a basic set of p-wavelets, then

$$\{(pN)^{j/2}\psi_l((pN)^j x \ominus \lambda) : 1 \leq l \leq pN-1,\ j \in \mathbb{Z},\ \lambda \in \Lambda, x \in R^+\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^+)$.

We denote $\psi_0 = \varphi$, the scaling function, and consider $pN-1$ functions ψ_l, $1 \leq l \leq pN-1$, in W_0 as possible candidates for wavelets. Since $\frac{1}{pN}\psi_l(\frac{\cdot}{pN}) \in V_{-1} \subset V_0$, it follows from property (iv) of definition (1.1) that for each l, $1 \leq l \leq pN-1$, there exist a sequence $\{a^l_\lambda : \lambda \in \Lambda\}$ with $\sum_{\lambda\in\Lambda} |a^l_\lambda|^2 < \infty$ such that

$$\frac{1}{pN}\psi_l(\frac{x}{pN}) = \sum_{\lambda\in\Lambda} a^l_\lambda \varphi(x \ominus \lambda). \tag{1.4}$$

Taking the Walsh-Fourier transform, we get

$$\hat{\psi}_l(pN\xi) = m_l(\xi)\hat{\varphi}(\xi), \tag{1.5}$$

where

$$m_l(\xi) = \sum_{\lambda\in\Lambda} a_\lambda^l \overline{\chi(\lambda,\xi)}. \tag{1.6}$$

The functions m_l, $0 \le l \le pN-1$, are in $L^2(\mathbb{R}^+)$. In view of the specific form of Λ, we observe that, $\Lambda = p\mathbb{Z} \cup (r/N + p\mathbb{Z})$, and so

$$m_l(\xi) = m_l^1(\xi) + \overline{\chi(r/N,\xi)} m_l^2(\xi), \quad 0 \le l \le pN-1, \tag{1.7}$$

where m_l^1 and m_l^2 are in $L^2(\mathbb{R}^+)$.

2. The Main Result

Let $\{V_j : j \in \mathbb{Z}\}$ be a nonuniform multiresolution p-analysis with scaling function φ. Then there exists a function m_0 such that $\hat{\varphi}(\xi) = m_0(\xi/pN)\hat{\varphi}(\xi/pN)$, where $m_0(\xi)$ is as in (1.7).

Applying the splitting lemma to the subspace V_1, we get the functions w_l, $0 \le l \le pN-1$, where

$$\hat{w}_l(\xi) = m_l(\xi/pN)\hat{\varphi}(\xi/pN), \tag{2.1}$$

such that $\{w_l(x \ominus \lambda) : 0 \le l \le pN-1,\ \lambda \in \Lambda,\ x \in R^+\}$ forms an orthonormal basis for V_1. Observes that $w_0 = \varphi$, the scaling function and w_l, $0 \le l \le pN-1$, are the basic p-wavelets.

We now define w_n for each integer $n \ge 0$. Suppose that for $s \ge 0$, w_s is already defined. Then define w_{q+pNs}, $0 \le q \le pN-1$, by

$$w_{q+pNs}(x) = \sum_{\lambda\in\Lambda} (pN) a_\lambda^q w_s(pNx - \lambda). \tag{2.2}$$

Note that (2.2) define w_n for all $n \ge 0$. Taking the Walsh-Fourier transform on both sides of (2.2), we get

$$(w_{q+pNs})^\wedge(\xi) = m_q(\xi/pN)\hat{w}_p(\xi/pN), \ \ 0 \le q \le pN-1. \tag{2.3}$$

The functions $\{w_n : n \ge 0\}$ will be called the basic nonuniform p-wavelet packets associated with nonuniform multiresolution p-analysis.

Theorem 2.1. *Let $\{w_n : n \geq 0\}$ be the basic nonuniform p-wavelet packets associated with the nonuniform multiresolution p-analysis $\{V_j\}$. Then*

$$\{w_n(x \ominus \lambda) : \ n \geq 0, \ \lambda \in \Lambda, \ x \in R^+\}$$

is an orthonormal basis of $L^2(\mathbb{R}^+)$.

Proof. First we prove that $\{w_n(x \ominus \lambda) : \ (pN)^j \leq n \leq (pN)^{j+1} - 1, \ \lambda \in \Lambda, \ x \in R^+\}$ is an orthonormal basis of $W_j, \ j \geq 0$ (by induction on j) . Since $\{w_n : 1 \leq n \leq pN - 1\}$ are the basic wavelets, (i) is true for $j = 0$. Assume that it holds for j. By (1.2) and the assumption, we have

$$\{(pN)^{1/2} w_n((pN)x \ominus \lambda) : \ (pN)^j \leq n \leq (pN)^{j+1} - 1, \ \lambda \in \Lambda, \ x \in R^+\}$$

is an orthonormal basis of W_{j+1}. Denote

$$E_n = \overline{span}\{(pN)^{1/2} w_n((pN)x \ominus \lambda) : \ \lambda \in \Lambda, \ x \in R^+\},$$

so that

$$W_{j+1} = \bigoplus_{n=(pN)^j}^{(pN)^{j+1}-1} E_n.$$

Now we get functions $g_l^{(n)}$, $0 \leq l \leq pN - 1$, defined by

$$(g_l^{(n)})^\wedge(\xi) = m_l(\xi/pN)\hat{w}_p(\xi/pN), \ \ 0 \leq l \leq pN - 1,$$

such that $\{g_l^{(n)}(x \ominus \lambda) : \ \ 0 \leq l \leq pN - 1, \lambda \in \Lambda, x \in R^+\}$ is an orthonormal basis of E_n, and we have the expansion as in the follows

$$(g_l^{(n)})^\wedge(\xi) = m_l(\xi/pN) m_{\mu_1}(\xi/(pN)^2) \cdots m_{\mu_j}(\xi/(pN)^{j+1}) \hat{\varphi}(\xi/(pN)^{j+1}).$$

But, the expression on the right-hand side is precisely $w_m(i)$, where

$$m = l + (pN)\mu_1 + (pN)^2\mu_2 + \cdots + (pN)^j \mu_j = l + pNn.$$

Hence, we get $g_l^{(n)} = w_{l+pNn}$. Since

$$\begin{aligned} &\{l + pNn : \ \ 0 \leq l \leq pN - 1, \ \ (pN)^j \leq n \leq (pN)^{j+1} - 1\} \\ &\quad = \{n : \ \ (pN)^{j+1} \leq n \leq (pN)^{j+2} - 1\}. \end{aligned}$$

Thus we have proved theorem for $j + 1$ and the induction is complete. Now because $V_j = V_0 \oplus W_0 \oplus \cdots \oplus W_{j-1}$ we have $\{w_n(x \ominus \lambda) : 0 \leq n \leq (pN)^j - 1, \ \lambda \in \Lambda, \ x \in R^+\}$ is an orthonormal basis of $V_j, \ j \geq 0$ and from the decomposition (1.3) the proof is complete. □

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