

A NEW APPROACH TO THE RANDOM RIEMANN SUM

Elham Dastranj

Shahrood University of Technology

P.O. Box. 203-2308889030, Shahrood, IRAN

Abstract: In the investigation of convergence of random Riemann sums to the Lebesgue integral, the underlying measure is the Lebesgue one. In this paper we generalize the Lebesgue measure to an atom less measure that is positive on any interval with positive length.

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1. Introduction

The concept of random Riemann sums is investigated in [1], [2], [3], [4], [5], [6] and is introduced in [4] and [5] in the following manner. Denote the interval $[0, 1)$ by I and let I be equipped with Borel σ -algebra. Let m be the Lebesgue measure on I . By a partition \mathcal{P}_0 of I we mean a finite sequence, x_0, x_1, \dots, x_n of elements of I such that $0 = x_0 < x_1 < \dots < x_n = 1, n \geq 1$. The norm of \mathcal{P}_0 with respect to the arbitrary measure μ on I is $\|\mathcal{P}_0\|_\mu := \max\{\mu(I_k) : I_k = [x_{k-1}, x_k) : 1 \leq k \leq n\}$. For each $I_k \in \mathcal{P}_0$, let $t_k \in I_k, 1 \leq k \leq n$, be a random variable with uniform distribution in the interval (x_{k-1}, x_k) , t_k 's being independent. Let $f : I \rightarrow \mathbb{R}$ be a Lebesgue integrable function. The random Riemann sum of f on \mathcal{P}_0 is

$$\mathcal{S}_{\mathcal{P}_0}(f) = \sum_{i=1}^n f(t_k)m(I_k).$$

In [4] some results are proved for Lebesgue measure m . As an example, Propo-

sition 2.1 of [4], can be mentioned to be equivalent to the following

Theorem 1. *For any $\epsilon > 0$, and any sequence of partitions $\mathcal{P}_n, n \geq 1$, of I whose elements are finite unions of disjoint intervals, if $\lim_{n \rightarrow \infty} |\mathcal{P}_n|_m = 0$, then*

$$P(|S_{\mathcal{P}_n}(f) - \int_I f dm| > \epsilon) \rightarrow 0.$$

In [2] the sequence of partitions based on which the random Riemann sums are defined is randomized. In this paper we generalize the random Riemann sums to an atom less measure that is positive on any interval with positive length. Throughout the paper μ is a fixed such measure. In fact we pose a condition for μ from which it follows that μ is atom less.

2. Random Riemann-Stieltjes Sums

For the partition \mathcal{P}_0 as described above, let z_k be a random element of $I_k \in \mathcal{P}_0$, chosen according to the probability law $\frac{\mu}{\mu(I_k)}$. Let $z_k, 1 \leq k \leq n$ be independent. Suppose $f : I \rightarrow \mathbb{R}$ is a μ -integrable function. We define a random Riemann-Stieltjes sum of f on \mathcal{P}_0 as follows:

$$S'_{\mathcal{P}_0}(f) = \sum f(z_k)\mu(I_k) = \sum f(z_k)[g(x_k) - g(x_{k-1})],$$

where $g(x) = \mu[0, x]$. It is a known fact that two Polish probability spaces whose probability measures are atom less are isomorphic. It follows that the spaces $([0, 1], \mathbf{B}, m)$ and $([0, 1], \mathbf{B}, \mu)$ are isomorphic. In the following theorem, we construct an isomorphism which is strictly increasing and hence continuous.

Theorem 2. *There is an isomorphic function $h : ([0, 1], B, m) \rightarrow ([0, 1], B, \mu)$ such that h is strictly increasing and hence continuous.*

Let $\{\mathcal{P}_n\}_{n \geq 1}$ be the sequence of partitions of I obtained by continued dissections, i.e.,

$$\mathcal{P}_1 = \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}, \mathcal{P}_2 = \{[0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4}), [\frac{3}{4}, 1)\}, \dots$$

For each $n \in \mathbb{N}$, let $x_1^{(n)}, x_2^{(n)}, \dots, x_{2^n}^{(n)} \neq 1$ be the elements in increasing order which constitute \mathcal{P}_n and $g(x_i^{(n)}) = y_i^{(n)}, 1 \leq i \leq 2^n$. Since μ is atom less, we

have $|\mathcal{P}_n|_{\mu} \rightarrow 0$, when $n \rightarrow \infty$. Since μ is positive on any interval with positive length, the set of $y_i^{(n)} : 1 \leq i \leq 2^n, n \geq 1$ is dense in I . For each $n \geq N$ and $1 \leq i \leq 2^n$ define $\varphi(y_i^{(n)}) = x_i^{(n)}$. It is clear that φ is a strictly increasing function and can uniquely be extended to a strictly increasing function from I to I . Now we take h be the extension of φ . Since the collection of intervals $[y_i^{(n)}, y_{i+1}^{(n)}], n \geq 1, 1 \leq i \leq 2^n$, constitutes a semi ring and generates the Borel σ -algebra, it follows that h is an isomorphic.

3. Main Results

Theorem 3. *If $\{\mathcal{P}_n\}_{n \geq 1}$ be a sequence of partitions of I such that $|\mathcal{P}_n|_{\mu} \rightarrow 0$, when $n \rightarrow \infty$, then*

$$P(|\mathcal{S}'_{\mathcal{P}_n}(f) - \int_I f d\mu| > \epsilon) \rightarrow 0.$$

Proof. Suppose h is the same as above. In view of the change of variable theorem and h being measure preserving

$$|\sum f(z_k)\mu(I_k) - \int_I f d\mu| > \epsilon \quad \text{and} \quad |\sum foh(t_k)m(h(I_k)) - \int_I fohdm| > \epsilon$$

have the same distribution. Hence in view of h being strictly increasing and Proposition 2.1 in [4], the truth of the result is obvious. □

Remark 1. It is clear that if μ is not atom less then the sequence $\{\mathcal{P}_n\}_{n \geq 1}$ in the above does not exist.

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