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CONSERVATION LAW MODEL OF SERIAL SUPPLY CHAIN NETWORK INCORPORATING VARIOUS VELOCITY FORMS

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Abstract: A mathematical model describing supply chain on a network is introduced. Due to inventories and various properties of the suppliers, it is not realistic to choose the velocity in supply chain to be constant in every problem. So the constitutive relationships for supply chain velocity are presented and combined with the non-smooth flux function of supply chain. Various examples relevant to real applications are presented through the characteristics approach and verified numerically. Numerical experiments are performed in a supply chain with three nodes. WIP in each node, flux and density of supply chain are analyzed.

AMS Subject Classification: 97Mxx, 03H10, 65C20
Key Words: supply chain, conservation laws, non-smooth flux, network, WIP, front-tracking

1. Introduction

In recent years, production flows in a supply chain network have become an important research prospect. A supply chain network can be considered as an organization of activities, that consists of suppliers, manufacturers, warehouses and customers. Here we will consider the material flow from suppliers to customers through various internal steps.

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The main problem in supply chain is to design the model and evaluate the performance of supply chains. Product variety, inventory decision, frequent introduction of new products, changes in customer demand, minimum cost, high quality etc. make the problem even more challenging and complicated.

Several mathematical approaches have been proposed for supply chain modeling. There are discrete event simulation (DES) models, based on considerations of individual parts in supply chain [1]. It is very powerful and common approach for the mathematical modeling in supply chain. DES provides an accurate description of supply chain dynamics. Here the idea is to track goods from supplier to customer. In mathematical sense, track means computing arrival times of each parts over the whole network.

On the other direction, continuous models using partial differential equations have been introduced and investigated during recent years (see [1], [4]-[10], [12], [14]). These models describe the evolution of flows, in particular flow of parts in a single supplier of a supply chain network. Inclusion of non-linearities in the dynamics provides great advantage of continuous model approach. Another main advantage for this models is that they are scale-invariant in the number of parts.

We consider a chain of M suppliers. Every supplier m receives a certain good (measured in units of parts) from supplier $m-1$, processes the material and passes to the next supplier $m+1$. Supplier m is characterized by its throughput time $T(m)$ and its maximal capacity $\mu(m)$. To compute the time evolution of each part in the supply chain, the modeling of the queues are essential. By assuming FIFO policy, the state of the queue will be either empty or non-empty. Whenever the queue is non-empty, the part has to wait and the waiting time is inverse of the processing rate. If $\tau(m,n)$ denotes the arrival time of part n to supplier m, then it leads to the following [1]:

$$\tau(m+1,n) = \max\{\tau(m,n) + T(m), \tau(m+1,n-1) + \frac{1}{\mu(m,n-1)}\}. \quad (1.1)$$

The above time recursion is analyzed using the Newell-curves (N-curves) as following [15]:

$$U(m,t) = \sum_{n=0}^{\infty} H(t - \tau(m,n)) = 1, ..., M, \quad t > 0. \quad (1.2)$$

The above discrete model provides necessary information to proceed in the direction of continuous modeling. Mapping each supplier onto one gridpoint in space, performing the asymptotic analysis (taking $M \to \infty$) and using the concept of virtual processors (decomposed each supplier into many virtual suppliers to validate in finite number of suppliers case) the time-recursion (1.1) can be approximated through the conservation law :

$$\frac{\partial \rho}{\partial t} + \frac{\partial (f(\rho))}{\partial x} = 0, \ \forall \ x \in [0,1], \ t > 0, \tag{1.3}$$

$$f(\rho) := \min\{\mu, v\rho\}, \tag{1.4}$$

where $\rho(x,t)$ and $f(\rho)$ denote the parts density and flux of parts respectively. $v(= \frac{1}{T})$ can be function of space and time. μ can be function of x and t. The above hyperbolic equation is non-linear due to the 'min' function in the flux.

It is very wise to choose initial values of densities are zero throughout the supply chain. In many real life situations, it is quite frequently seen that some materials wait initially inside the supply chain. They wait until the proper amount of other materials come. Then the processor starts the process. Therefore, it is very significant to study the situation of the suppliers after the processing starts in supply chain. We analyze this fact through the characteristics approach after time $t > 0$.

In this paper, we restrict our interest to deterministic PDE modeling of supply chain and its efficient solutions. However, the PDE models allow us to focus on simulations driven by time-dependent influx $\lambda(t)$. The aim of this paper is to introduce different types of velocities in supply chain with non-smooth flux and it's rigorous analysis.

The paper is organized as follows. In Section 2, we formulate the continuum modeling equation for supply chain networks. First we consider a single node in a supply chain network and establish the model by considering various constitutive equations of velocity. We then extend this formulation to a serial supply chain network. In Section 3, we analyze the solution procedure of the model through the characteristics approach. Section 4 is devoted to numerical results and discussion. We discuss about the applicability of the model in Section 5.

2. A Model for Supply Chain

Let us consider a single node (supplier) in the supply chain. Each supplier consists of a processor and a queue in front of it. If queue is empty, the material will directly go to the processor; else the material has to wait. In this aspect, we focus on continuum modeling now onwards in this paper. Since we mapped each supplier onto one grid point in space, it is reasonable to choose x as a continuous variable representing the completion of the product within the supplier. Parts at $x = 0$ represents raw material entered into the supplier and parts at $x = 1$ are the finished products going out of the supplier. Let $\rho(x,t)$ be the density of parts at stage x and time t. We denote μ as maximal processing rate and $v(x,t)$ represents the velocity of the product moving in the supplier. Let $y_l(x,t)$ denotes the yield-loss in the supply chain at stage x and time t, which can be considered as a function of parts density: $y_l = y_l(\rho)$. If $\lambda(t)$ is the arrival rate in the supply chain, then the following model can be obtained from the conservation of parts within the supplier:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (f(\rho))}{\partial x} + y_l(\rho) = 0, \quad \forall\, x \in [0,1],\ t > 0, \tag{2.1}$$

$$f(\rho) := \min\{\mu(x,t), v(x,t)\rho(x,t)\}. \tag{2.2}$$

Initial condition:

$$\rho(x,0) = \rho_0(x) \tag{2.3}$$

Influx condition :

$$f(0,t) = \lambda(t). \tag{2.4}$$

In the above model P appears as processing time for a part when it enter into the processor. Now the question arises: what will be the form of velocity in the supply chain $v(x,t)$? Before answering this question, we want to introduce the term 'Work in Progress', denoted by $WIP(t)$. From [1] we know the relation of WIP and density as following:

$$WIP(t) = \int_0^1 \rho(x,t)dx.$$

Let us assume that v is the function of part density. $v = v(\rho)$ is the equation relating the speed of the product moving through the supplier to the amount of product in the supplier, i.e. WIP. Therefore we have the form of velocity and that is $v = v(WIP(t))$. Sun et al. (2008) have described different forms of velocity. In this paper we will discuss the following forms of velocity:

$$v(\rho) = \frac{1}{P + P\rho(1,t)},$$
$$v(\rho) = \frac{1}{P + P.WIP(t)}.$$

Each velocity form combined with equations (2.1)-(2.4) represents the dynamics of the supplier.

2.1. Towards Network Modeling

In this section, we want to extend our model of single node to a serial supply chain. First we define the network in supply chain [9]. A supply chain network is a finite, connected, directed, simple graph consisting of arcs $A = 1, 2, ..., n$ and vertices $V = 1, 2, ..., n-1$. Each supplier is modeled by an arc j. Each arc again parameterized by an interval $[a_j, b_j]$.

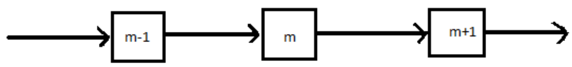

Figure 1: Serial Supply Chain

Let us consider a chain of M suppliers. Every supplier m received parts from supplier $m-1$ then it get processed and pass it to the next supplier $m+1$. We assume that the materials flow from node $m-1$ to m, and then m to $m+1$ and so on. Finally the product exit from supply chain at node M. Let $\rho_j(x,t)$, $j = 1, 2, ..., M$ denote the density of parts at node j, at stage x and at time t.

If $y_l(\rho_j)$ yield loss at supplier j, then conservation of materials can be presented as follows:

$$\frac{\partial \rho_j}{\partial t} + \frac{\partial (f_j(\rho))}{\partial x} + y_l(\rho_j) = 0, \quad t > 0, \quad x \in (0,1], \qquad j = 1, 2, ..., M,$$

$$f_j(\rho) := \min\{\mu_j(t), v_j \rho\}.$$

The velocity v_j of the product movement through the node j is a function of ρ_j.

$$v_j = v_j(\rho), \qquad j = 1, 2, ..., M.$$

We impose the following initial conditions for the local densities as following:

$$\rho_j(x,0) = \rho_{j,0}(x), \qquad j = 1, 2, ..., M.$$

Here we need to be careful regarding the influx condition. The first node need to be treated separately. We impose an influx which is eventually influx of supply chain, for the first node:

$$f_1(0,t) = \lambda_1(t).$$

In supply chain, we often need to provide some extra materials from outside or some unusable materials from the supplier need to be removed (quite frequently occur in fruits, vegetables supply network). So to improve the model as more efficient, we introduce the term λ_j. In the above model λ_j represents the extra or removed materials in node j of the supply chain. If we consider the equation of queue in the supplier in separate way, then the queue equation will be of following form. Each queue is a time dependent function $t \mapsto q_j(t)$ and used to buffer demands for the processor j. The inter-connection between the nodes give the following influx conditions for the remaining supplier j. Here two cases have to be considered as either queue is empty or the queue is non-empty. The following equation take care of both the circumstances.

$$\frac{dq_j}{dt} = \lambda_j(t) + f_{j-1}(1,t) - f_j(0,t), \quad q_j(0) = q_{j,0},$$

where $f_j(0,t)$ is defined as follows:

For $j = 2, 3, ..., M$

$$f_j(0,t) = \begin{cases} \min\{\lambda_j(t) + f_{j-1}(1,t), \mu_j(t)\} & \text{if } q_j(t) = 0 \\ \mu_j(t) & \text{if } q_j(t) \neq 0. \end{cases}$$

In this case, we have $v_j = v(W_j(t))$, where $W_j(t) = \int_0^1 \rho_j(x,t)dx$.

By introducing the connectivity matrix, we can model the supply chain having multiple of nodes like a network. The idea of connectivity matrix has been introduced by Sun [16]. Here the same way we can extend the model of a serial supply chain to the network model of complex geometries.

3. Solution Procedure

Here we use the idea of constructing solution $\rho(x,t)$ for all times t is given by front-tracking algorithm ([3],[11]). We basically start with a step function $\rho_0(x)$ and solve at each point of a jump discontinuity a Riemann problem. The evaluated solution $\rho(x,t)$, $t > 0$ is again a step function with discontinuities traveling at constant speed. Since the flux function of our problem is continuous and piecewise linear, all discontinuities of the solution will be referred as fronts. There is possibility that after some time one or more fronts may collide. Then we proceed solving Riemann problem with initial data at collision time. If we are able to show that the number of collisions is finite then for all times t, this procedure is well-defined and generates a solution $\rho(x,t)$.

We want to show that the number of interactions between the discontinuities are finite. Let $F(t)$ be the total number of fronts in the front tracking solution $\rho(x,t)$ at time t and $L(t)$ be the total number of linear segments present in all fronts $\rho(x,t)$ at times t. The number of linear segments in the piecewise linear flux denoted by S. We define $I(t) = SL(t) + F(t)$. We will show that $I(t)$ is strictly decreasing for every shock collision. Since flux function have $S-1$ breakpoints, after one collision $F(t)$ can be increasing atmost by $S-1$. But at the same time $L(t)$ decreases by 1. Now $S(L-1)+F+S-1 = SL+F-1 < I(t)$. Therefore, the number of collisions will be finite. Furthermore, by the construction of the solution, total variation is non-decreasing and bounded. We will illustrate some of the situations occurred in the characteristics analysis.

In Example 1, we discuss the situation when initial values of density have a single discontinuity. We analyze the nature after small time considering the initial values of the density having multiple discontinuities in Example 2. As we mentioned earlier, these types of examples occurred frequently since some of the materials initially stored in the suppliers. It starts processing after the materials come from the previous supplier. So, the following problems are quite relevant in supply chain point of view.

Example 1. Let us consider equation (2.1) with 100 percent yield. We have the constant (or piecewise constant) influx. We consider the initial condition: $\rho_0(x) = \frac{1}{4}$ for $x < \frac{1}{2}$ and $\rho_0(x) = 1$ for $x > \frac{1}{2}$. Let $Df(\rho)$ represents weak derivative of flux $f(\rho)$. Let us take $\mu = 1$ and $v = 2$. Now we have the following: $Df(\rho) = 2$ for $0 < x \leq \frac{1}{2}$ and $Df(\rho) = 0$ for $\frac{1}{2} < x \leq 1$. Let s be the speed of the discontinuity. Using the Rankine-Hugoniot condition, we get $s = \frac{2}{3}$. The solution of the discussed problem is following:

$$\rho(x,t) = \begin{cases} \frac{\lambda(t)}{v} & \text{for } x \leq t \\ \frac{1}{4} & \text{for } t < x \leq \frac{2}{3}t + \frac{1}{2} \\ 1 & \text{for } x > \frac{2}{3}t + \frac{1}{2}. \end{cases}$$

Since there is no more collision, the above solution holds for all time $t > 0$.

Example 2. Here we introduce the problem which brings an interesting situation as breakpoint in flux and shock waves (creating due to the different characteristics velocities) will interact. Let us consider the following initial data for density: $\rho_0(x) = 2$ for $x < \frac{1}{3}$, $\rho_0(x) = \frac{1}{4}$ for $\frac{1}{3} < x \leq \frac{2}{3}$ and $\rho_0(x) = 1$ for $x > \frac{2}{3}$ with conservation law (2.1) (assume 100 percent yield). We consider μ and v are same as previous example. Weak derivative of the flux $Df(\rho)$ is given by: $Df(\rho) = 0$ for $0 < x \leq \frac{1}{3}$, $Df(\rho) = 2$ for $\frac{1}{3} < x \leq \frac{2}{3}$ and $Df(\rho) = 0$

for $\frac{2}{3} < x \leq 1$. We can observe that $f(\rho)$ has a breakpoint at $\rho = \frac{1}{2}$. Here we need to consider two Riemann problems with initial values $(\rho_l, \rho_r) = \left(2, \frac{1}{2}\right)$ and $(\rho_l, \rho_r) = \left(\frac{1}{2}, \frac{1}{4}\right)$. Let s_1, s_2 and s_3 be the speed of the discontinuities respectively, then $s_1 = 0, s_2 = 2$ and $s_3 = \frac{2}{3}$. After sufficiently small time $t > 0$, we get the following solution:

$$\rho(x,t) = \begin{cases} 2 & \text{for } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{2} & \text{for } \frac{1}{3} < x \leq \frac{1}{3} + 2t \\ \frac{1}{4} & \text{for } \frac{1}{3} + 2t < x \leq \frac{2}{3} + \frac{2}{3}t \\ 1 & \text{for } x \geq \frac{2}{3} + \frac{2}{3}t. \end{cases}$$

The discontinuities at $x = \frac{1}{3}$ and $x = \frac{2}{3}$ will collide after time $t = \frac{1}{4}$ and create shock wave. So the above solution is no longer valid. The speed of the discontinuity will be $s_4 = 0$. After time $t = \frac{1}{4}$, the solution will be

$$\rho(x,t) = \begin{cases} 2 & \text{for } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{2} & \text{for } \frac{1}{3} < x \leq \frac{5}{6} \\ 1 & \text{for } \frac{5}{6} < x \leq 1. \end{cases}$$

Since there are no more collision, the above solution will be valid for all time $t > 0$. This solution will be verified numerically in the later section.

The above analysis can be carried out for different values of velocity form and for non-constant $\mu(x,t)$. The situation will be same and can be verified by the procedure discussed above. The solution can be constructed through either shock or discontinuities. Now we proceed to the numerical direction to validate the above solutions and to simulate the supply chain model in the next section.

4. Numerical Simulation

In this section, we focus on the serial supply chain. The case of supply chain network is similar to the serial supply chain and can be discussed the same way. We want to verify the Lipschitz continuity of our flux function. If $v(x,t)$ is constant, then it is easy to verify that $f(\rho)$ is Lipschitz continuous. If $v(x,t)$ is not constant, then also from our choice of velocity we can ensure that $\|v\|_\infty \leq K$, for some constant K. Since the flux function is Lipschitz, then any monotone, conservative and consistent scheme converges to unique entropy solution (for more, one can refer [13]).

Numerically, we discretize the nonlinear hyperbolic equation using an Upwind scheme [4]. We use trapezoidal rule to find the work in progress. For each supplier j, we use the following notation:

- L_j = length of the supplier j
- $\Delta x_j = \frac{L_j}{N_j}$, N_j = number of segments in discretization of supplier j
- $\Delta t_j = \frac{T}{M_j}$, M_j = number of segments in descretization of time $[0, T]$
- $\lambda(j)$ = influx at t_j
- $\rho(j, n)$ = approximate density of supplier j at point (x_j, t_n)
- $v(j, n)$ = approximate velocity at the point (x_j, t_n).

The upwind method reads as

$$\rho(j, n+1) = \rho(j, n) - \frac{\Delta t_j}{\Delta x_j}[f(j+1, n) - f(j, n)],$$

where

$$f(j, n) = \begin{cases} \lambda(n) & \text{for first supplier} \\ \min\{v(j-1, n)\rho(j-1, n), \mu(j-1, n)\} & \text{else} \end{cases}$$

with the CFL condition given by

$$\frac{\Delta t}{\Delta x}\|v\|_\infty < 1.$$

We first validate the solution of Problem-2 with the characteristics solution in this section. The step length in space and time length are considered in such a way that CFL condition will be satisfied. In Fig.2, we present the scenario after some short time starting with initial condition given in Problem-2. Clearly, the characteristics solution and numerical solution are the same.

In this section we conduct two numerical experiments with different form of velocity in the supply chain. We consider a supply chain with three nodes. We consider the length of X (the degree of completion interval) equal to unity. The nodes will represent the intervals [0,0.2], [0.2,0.8] and [0.8,1]. The maximal processing rate μ is given as follows, see [1]:

$$\mu(x, t) = \begin{cases} 15 & \text{for } 0 \le x \le 0.2 \\ 10 & \text{for } 0.2 < x \le 0.8 \\ 15 & \text{for } 0.8 < x \le 1. \end{cases}$$

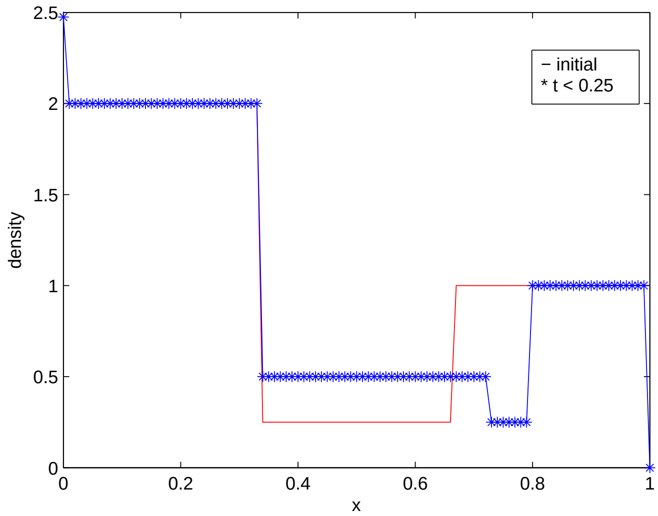

Figure 2: Solution with discontinuous initial data

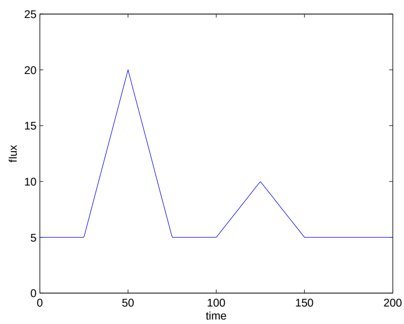

Figure 3: Influx in supply chain. It imposed on the left of the problem domain.

The given influx is displayed in Fig.3. First, we consider the following form of velocity:

$$v(\rho) = \frac{1}{P + P\rho(1,t)}.$$

Take the step length in space $\Delta x = 0.05$ and time length $\Delta t = 0.01$. Processing rate is taken as 0.15. WIP can be computed as $\int_a^b \rho(x,t)dx$ for each node. The

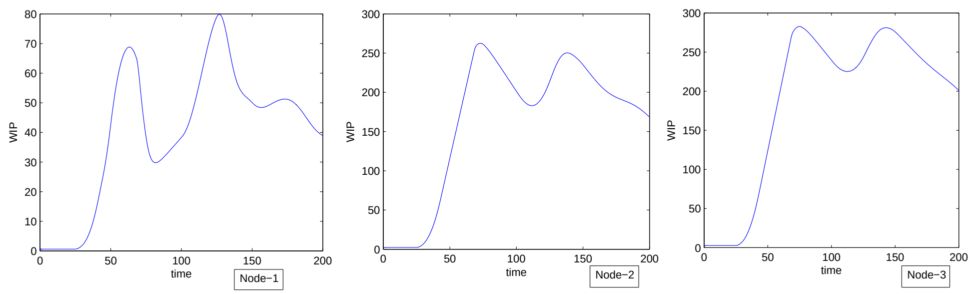

Figure 4: Work in progress

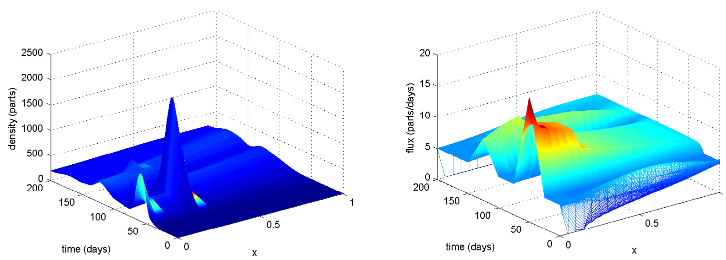

Figure 5: Density ρ and flux f in supply chain

work in progress (WIP) of each nodes are shown in Fig.4. The density and flux supply chain are presented in Fig.5. WIP is the critical information in

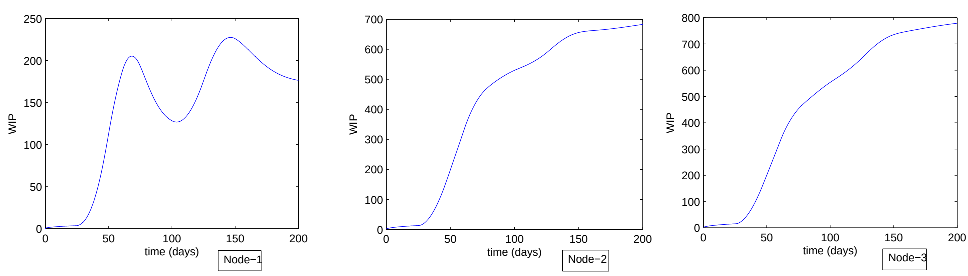

Figure 6: Work in progress

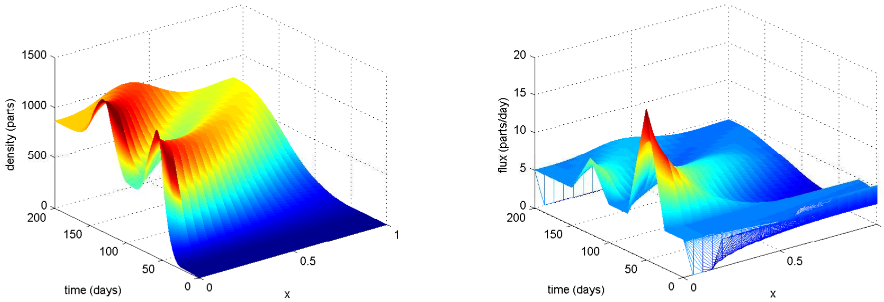

Figure 7: Density and flux in supply chain

supply chain management, in particular for inventory control and utilization management at each node. Accumulation of WIP and its effect to the flow are quite evident as displayed in the plots of WIP. As expected, bottlenecks occurred at $x = 0$ and $x = 0.2$.

Now we consider the following form of velocity:

$$v(\rho) = \frac{1}{P + P.WIP(t)}.$$

Here the maximal processing rate and nodes in supply chain are considered same as above. The processing rate is taken as 0.2. We discretize the conservation law using Upwind scheme and evaluating WIP using trapezoidal rule. The work in progress for each node is shown in Fig.6. The flux and density in supply chain are presented in Fig.7.

5. Applications

In this section, we will clearly specify the applicability of the presented model and the corresponding numerical simulation in the supply chain network. The model enriches the study of PDE based supply chain modeling in many ways.

- It is seen in many real supply chain problems like vegetables, processing food items or electronics commodities, we can not start from an empty supply chain. There are some initial materials should be available to some particular suppliers. One can easily realize that the provided initial materials may not come continuously as expected. The presented model analyze the situation and provided the corresponding density distribution in the supply chain.

- It is not appropriate to choose the speed of the materials as constant. It basically depends on the density in the supply chain. Whenever the density increases, velocity should decrease. The forms of the velocity considered in this paper definitely reflect that. So, it is quite evident that the presented model is much more suggestible than the existing models as far as application is concern.
- The presented model for network have the flexibility that some materials can be entered into the chain from outside before the processor. This is an important aspect for various supply chain network. After the processing starts, there may be need for some materials in supplier m at time $t = t_0$ which to be provided from outside. The situation is incorporated by introducing a time dependent term in the flux function.
- The presented numerical simulation will help to analyze the overview of a serial supply chain. The occurrence of bottlenecks, outflux are some of the aspects.

6. Conclusion

In a real life supply chain, the velocity may not be constant throughout. So, we present a continuous model with non-smooth flux attached with the various form of velocities. Afterwards, based on the proposed models, some numerical examples have been presented. The proposed model can be used to obtain more accurate results for material flows in supply chain networks from macroscopic perspective. In various scenario our models can be applicable.

We would like to concentrate on priority based models which are more relevant with real situation. Also the yield-loss in supply chain prospective is on process. Moreover, optimization perspective like maximum outflux and minimum costs are also be desired for the proposed models. We can expect some important features like re-entrant nodes and risk analysis of supply chain networks to be introduced by the proposed continuum modeling method.

Acknowledgments

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