International Journal of Applied Mathematics

Volume 26 No. 2 2013, 221-231

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v26i2.8

AN L^p INEQUALITY FOR POLYNOMIALS NOT VANISHING IN A DISK

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Abstract: For the class of polynomials $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, of degree n not vanishing in the disk $|z| < k$ where $k \geq 1$, we investigate the dependence of $\|P(Rz) - P(rz)\|_p$ on $\|P(z)\|_p$ for $R > r \geq 1$, $p > 0$ and present compact generalizations of certain well-known polynomial inequalities.

AMS Subject Classification: 30C10, 30A10, 41A17
Key Words: polynomials, inequalities in the complex domain, Zygmund's inequality

1. Introduction and Statement of Results

Let $\mathscr{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree n. For $P \in \mathscr{P}_n$, define

$$\|P(z)\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p \right\}^{1/p}, \ 0 < p < \infty,$$

$$\|P(z)\|_\infty := \max_{|z|=1} |P(z)| \quad \text{and} \quad m(P,k) = \min_{|z|=k} |P(z)|.$$

Received: March 29, 2013

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If $P \in \mathscr{P}_n$, then

$$\left\|P'(z)\right\|_p \leq n \left\|P(z)\right\|_p, \quad p \geq 1 \tag{1.1}$$

and

$$\left\|P(Rz)\right\|_p \leq R^n \left\|P(z)\right\|_p, \ \ R > 1, \ \ p > 0. \tag{1.2}$$

The inequality (1.1) was found by Zygmund [18] whereas inequality (1.2) is a simple consequence of a result of Hardy [10]. Arestov [2] proved that (1.1) remains true for $0 < p < 1$ as well. For $p = \infty$, the inequality (1.1) is due to Bernstein (for reference, see [13, 16, 17]) whereas the case $p = \infty$ of inequality (1.2) is a simple consequence of the maximum modulus principle (see [13, 14, 16]). Both inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zero in $|z| < 1$. In fact, if $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then inequalities (1.1) and (1.2) can be respectively replaced by

$$\left\|P'(z)\right\|_p \leq n \frac{\left\|P(z)\right\|_p}{\left\|1+z\right\|_p}, \ \ p > 0 \tag{1.3}$$

and

$$\left\|P(Rz)\right\|_p \leq \frac{\left\|R^n z + 1\right\|_p}{\left\|1+z\right\|_p} \left\|P(z)\right\|_p, \ \ R > 1, \ \ p > 0. \tag{1.4}$$

Inequality (1.3) is due to De-Bruijn [9] (see also [3]) for $p \geq 1$. Rahman and Schmeisser [15] extended it for $0 < p < 1$ whereas the inequality (1.4) was proved by Boas and Rahman [8] for $p \geq 1$ and later it was extended for $0 < p < 1$ by Rahman and Schmeisser [15]. For $p = \infty$, the inequality (1.3) was conjectured by Erdös and later verified by Lax [11] whereas inequality (1.4) was proved by Ankeny and Rivlin [1].

As a compact generalization of inequalities (1.1) and (1.2), Aziz and Rather [6] proved that if $P \in \mathscr{P}_n$, then for every real or complex number α with $|\alpha| \leq 1$, $R \geq 1$, and $p > 0$,

$$\left\|P(Rz) - \alpha P(z)\right\|_p \leq \left|R^n - \alpha\right| \left\|P(z)\right\|_p, \tag{1.5}$$

and if $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R \geq 1$, and $p > 0$,

$$\left\|P(Rz) - \alpha P(z)\right\|_p \leq \frac{\left\|(R^n - \alpha)z + (1-\alpha)\right\|_p}{\left\|1+z\right\|_p} \left\|P(z)\right\|_p. \tag{1.6}$$

The inequality (1.6) is the corresponding compact generalization of inequalities (1.3) and (1.4).

Recently, A. Aziz and Q. Aliya [4] considered, for a fixed μ, the class of polynomials

$$\mathscr{P}_{n,\mu} := \left(P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, \quad 1 \le \mu \le n\right)$$

of degree at most n not vanishing in the disk $|z| < k$ where $k \ge 1$ and investigated the dependence of

$$\big\|P(Rz) - P(rz)\big\|_\infty \quad \text{on} \quad \|P(z)\|_\infty, \quad m(P,k),$$

and proved that if $P \in \mathscr{P}_{n,\mu}$ and $P(z)$ does not vanish for $|z| < k$ where $k \ge 1$ then for every $R > r \ge 1$, $0 \le t \le 1$ and $|z| = 1$,

$$\big|P(Rz) - P(rz)\big| \le \frac{R^n - r^n}{1 + k^\mu \phi(R,r,\mu,k)}\Big(\big\|P(z)\big\|_\infty - tm(P,k)\Big), \tag{1.7}$$

where

$$\phi(R,r,\mu,k) := \frac{k + \lambda(R,r,\mu,k)}{1 + k\lambda(R,r,\mu,k)} \ge 1 \tag{1.8}$$

and

$$\lambda(R,r,\mu,k) := \left(\frac{R^\mu - r^\mu}{R^n - r^n}\right)\left(\frac{|a_\mu| k^n}{|a_0| - t\, m(P,k)}\right) \le 1.$$

In this paper, we establish L^p-mean extensions of inequality (1.7) for $0 < p < \infty$. More precisely, we prove:

Theorem 1. *If $P \in \mathscr{P}_{n,\mu}$ and $P(z)$ does not vanish for $|z| < k$ where $k \ge 1$, then for $\delta \in \mathbb{C}$ with $|\delta| \le 1$, $0 < p < \infty$, $0 \le t \le 1$ and $R > r \ge 1$,*

$$\left\|P(Rz) - P(rz) + \delta t\left\{\frac{R^n - r^n}{1 + k^\mu \phi(R,r,\mu,k)}\right\} m(P,k)\right\|_p \le \frac{R^n - r^n}{\big\|k^\mu \phi(R,r,\mu,k) + z\big\|_p}\big\|P(z)\big\|_p, \tag{1.9}$$

where $\phi(R,r,\mu,k)$ is defined by (1.8).

Remark 1. If we let $p \to \infty$ in inequality (1.9) and choose argument of δ suitably with $|\delta| \to 1$, we get inequality (1.7).

Taking $t = 0$ in (1.9), we obtain the following result.

Corollary 1. *If $P \in \mathscr{P}_{n,\mu}$ and $P(z)$ does not vanish for $|z| < k$ where $k \geq 1$, then for $\delta \in \mathbb{C}$ with $|\delta| \leq 1$, $0 < p < \infty$ and $R > r \geq 1$,*

$$\left\|P(Rz) - P(rz)\right\|_p \leq \frac{R^n - r^n}{\left\|k^\mu \phi(R, r, \mu, k) + z\right\|_p} \left\|P(z)\right\|_p, \tag{1.10}$$

where $\phi(R, r, \mu, k)$ is defined by (1.8).

If we divide the two sides of inequality (1.9) by $R - r$ and letting $R \to r$, we get the following result.

Corollary 2. *If $P \in \mathscr{P}_{n,\mu}$ and $P(z)$ does not vanish for $|z| < k$ where $k \geq 1$, then for $\delta \in \mathbb{C}$ with $|\delta| \leq 1$, $0 < p < \infty$, $0 \leq t \leq 1$ and $R > r \geq 1$,*

$$\left\|zP'(rz) + \delta t \left\{\frac{nr^{n-1}}{1 + k^\mu \psi(r, \mu, k)}\right\} m(P, k)\right\|_p \leq \frac{nr^{n-1}}{\left\|k^\mu \psi(r, \mu, k) + z\right\|_p} \left\|P(z)\right\|_p, \tag{1.11}$$

where

$$\psi(r, \mu, k) := \frac{k + \frac{\mu}{nr^{n-\mu}} \left(\frac{|a_\mu| k^n}{|a_0| - t\, m(P,k)}\right)}{1 + k \frac{\mu}{nr^{n-\mu}} \left(\frac{|a_\mu| k^n}{|a_0| - t\, m(P,k)}\right)}. \tag{1.12}$$

For $k = 1$ and $t = 0$ inequality (1.11) reduces to inequality (1.3) for $p > 0$.

By using Minkowski's inequality, we obtain from (1.9), for $p \geq 1$,

$$\begin{aligned}
&\left\|P(Rz) + \delta t\left\{\frac{R^n - r^n}{1 + k^\mu \phi(R, r, \mu, k)}\right\}m(P, k)\right\|_p \\
&= \left\|P(Rz) - P(rz) + \delta t\left\{\frac{R^n - r^n}{1 + k^\mu \phi(R, r, \mu, k)}\right\}m(P, k) - P(rz)\right\|_p \\
&\leq \left\|P(Rz) - P(rz) + \delta t\left\{\frac{R^n - r^n}{1 + k^\mu \phi(R, r, \mu, k)}\right\}m(P, k)\right\|_p + \left\|P(rz)\right\|_p \\
&\leq \frac{R^n - r^n}{\left\|k^\mu \phi(R, r, \mu, k) + z\right\|_p}\left\|P(z)\right\|_p + \left\|P(rz)\right\|_p. \qquad (1.13)
\end{aligned}$$

Inequality (1.13) in conjunction with inequality (1.4) gives the following result.

Corollary 3. *If $P \in \mathscr{P}_{n,\mu}$ and $P(z)$ does not vanish for $|z| < k$ where $k \geq 1$, then for $\delta \in \mathbb{C}$ with $|\delta| \leq 1$, $1 \leq p < \infty$, $0 \leq t \leq 1$ and $R > r \geq 1$,*

$$\begin{aligned}
&\left\|P(Rz) + \delta t\left\{\frac{R^n - r^n}{1 + k^\mu \phi(R, r, \mu, k)}\right\}m(P, k)\right\|_p \\
&\qquad \leq \left\{\frac{R^n - r^n}{\left\|k^\mu \phi(R, r, \mu, k) + z\right\|_p} + \frac{\|r^n z + 1\|_p}{\|1 + z\|_p}\right\}\left\|P(z)\right\|_p, \qquad (1.14)
\end{aligned}$$

where $\phi(R, r, \mu, k)$ is defined by (1.8).

Letting $R \to r$ in (1.14), we obtain the following result.

Corollary 4. *If $P \in \mathscr{P}_{n,\mu}$ and $P(z)$ does not vanish for $|z| < k$ where $k \geq 1$, then for $\delta \in \mathbb{C}$ with $|\delta| \leq 1$, $1 \leq p < \infty$, $0 \leq t \leq 1$ and $r \geq 1$,*

$$\begin{aligned}
&\left\|P(rz) + \delta t\left\{\frac{nr^{n-1}}{1 + k^\mu \psi(r, \mu, k)}\right\}m(P, k)\right\|_p \\
&\qquad \leq \left\{\frac{nr^{n-1}}{\left\|k^\mu \psi(r, \mu, k) + z\right\|_p} + \frac{\|r^n z + 1\|_p}{\|1 + z\|_p}\right\}\left\|P(z)\right\|_p, \qquad (1.15)
\end{aligned}$$

where $\psi(r, \mu, k)$ is defined by (1.12).

2. Lemmas

To prove the above theorem, we need the following lemmas. The first lemma is due to Aziz and Aliya [4].

Lemma 1. *If $P \in \mathscr{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \geq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$, then for $R \geq r \geq 1$, $0 \leq t \leq 1$ and $|z| = 1$,*

$$\begin{aligned} k^{\mu}\phi(R, r, \mu, k)\big|P(Rz) - P(rz)\big| \\ \leq \big|Q(Rz) - Q(rz)\big| - t(R^n - r^n)m(P, k)\,, \end{aligned} \tag{2.1}$$

where $\phi(R, r, \mu, k)$ is given by (1.8).

The following lemma is a special case of result due to Aziz and Rather [7, Lemma 4].

Lemma 2. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every $p > 0$, $R > r \geq 1$ and for γ real, $0 \leq \gamma < 2\pi$,*

$$\begin{aligned} \int_0^{2\pi} \Big|\big(P(Re^{i\theta}) - P(re^{i\theta})\big) + e^{i\gamma}\big(R^n P(e^{i\theta}/R) - r^n P(e^{i\theta}/r)\big)\Big|^p d\theta \\ \leq \big(R^n - r^n\big)^p \int_0^{2\pi} \Big|P(e^{i\theta})\Big|^p d\theta. \end{aligned} \tag{2.2}$$

We also need the following lemma [5].

Lemma 3. *If A, B, C are non-negative real numbers such that $B + C \leq A$, then for each real number γ,*

$$|(A - C)e^{i\gamma} + (B + C)| \leq |Ae^{i\gamma} + B|.$$

3. Proof of Theorem

Proof of Theorem 1. By hypothesis, $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, therefore by Lemma 1, we have

$$\begin{aligned} k^{\mu}\phi(R, r, \mu, k)\big|P(Rz) - P(rz)\big| \\ \leq \big|Q(Rz) - Q(rz)\big| - t(R^n - r^n)m(P, k) \end{aligned}$$

for $|z|=1$, and $R>r\geq 1$ where $Q(z)=z^n\overline{P(1/\overline{z})}$. Equivalently,

$$k^\mu\phi(R,r,\mu,k)\big|P(Rz)-P(rz)\big| \\ \leq \big|R^nP(z/R)-r^nP(z/r)\big|-t(R^n-r^n)m(P,k)$$

for $|z|=1$, and $R>r\geq 1$. This inequality can be written as

$$k^\mu\phi(R,r,\mu,k)\left[\big|P(Rz)-P(rz)\big|+\frac{t(R^n-r^n)}{1+k^\mu\phi(R,r,\mu,k)}m(P,k)\right] \\ \leq \big|R^nP(z/R)-r^nP(z/r)\big|-\frac{t(R^n-r^n)}{1+k^\mu\phi(R,r,\mu,k)}m(P,k) \tag{3.1}$$

for $|z|=1$. Taking

$$A=\big|R^nP(z/R)-r^nP(z/r)\big|,\quad B=\big|P(Rz)-P(rz)\big|$$

and

$$C=\frac{t(R^n-r^n)}{1+k^\mu\phi(R,r,\mu,k)}m(P,k)$$

in Lemma 3 and noting by (1.8) and (3.1) that

$$B+C\leq A-C\leq A,$$

we get for every real γ,

$$\left|\left\{\big|R^nP(e^{i\theta}/R)-r^nP(e^{i\theta}/r)\big|-\frac{t(R^n-r^n)}{1+k^\mu\phi(R,r,\mu,k)}m(P,k)\right\}e^{i\gamma}\right. \\ \left.+\left\{\big|P(Re^{i\theta})-P(re^{i\theta})\big|+\frac{t(R^n-r^n)}{1+k^\mu\phi(R,r,\mu,k)}m(P,k)\right\}\right| \\ \leq\Big|\big|R^nP(e^{i\theta}/R)-r^nP(e^{i\theta}/r)\big|e^{i\gamma}+\big|P(Re^{i\theta})-P(re^{i\theta})\big|\Big|.$$

This implies for each $p>0$,

$$\int_0^{2\pi}\big|F(\theta)+e^{i\gamma}G(\theta)\big|^pd\theta \\ \leq\int_o^{2\pi}\Big|\big|R^nP(e^{i\theta}/R)-r^nP(e^{i\theta}/r)\big|e^{i\gamma}+\big|P(Re^{i\theta})-P(re^{i\theta})\big|\Big|^pd\theta, \tag{3.2}$$

where

$$F(\theta) = \left|P(Re^{i\theta}) - P(re^{i\theta})\right| + \frac{t(R^n - r^n)}{1 + k^\mu \phi(R, r, \mu, k)} m(P, k) \quad \text{and}$$

$$G(\theta) = \left|R^n P(e^{i\theta}/R) - r^n P(e^{i\theta}/r)\right| - \frac{t(R^n - r^n)}{1 + k^\mu \phi(R, r, \mu, k)} m(P, k).$$

Integrating both sides of (3.2) with respect to γ from 0 to 2π, we get with the help of Lemma 2 for each $p > 0$,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \left|F(\theta) + e^{i\gamma}G(\theta)\right|^p d\theta d\gamma \\
&\leq \int_0^{2\pi}\int_o^{2\pi} \left|\left|R^n P(e^{i\theta}/R) - r^n P(e^{i\theta}/r)\right|e^{i\gamma} + \left|P(Re^{i\theta}) - P(re^{i\theta})\right|\right|^p d\theta d\gamma \\
&= \int_0^{2\pi}\left\{\int_o^{2\pi} \left|\left|R^n P(e^{i\theta}/R) - r^n P(e^{i\theta}/r)\right|e^{i\gamma} + \left|P(Re^{i\theta}) - P(re^{i\theta})\right|\right|^p d\gamma\right\} d\theta \\
&= \int_0^{2\pi}\left\{\int_o^{2\pi} \left|\left(R^n P(e^{i\theta}/R) - r^n P(e^{i\theta}/r)\right)e^{i\gamma} + \left(P(Re^{i\theta}) - P(re^{i\theta})\right)\right|^p d\gamma\right\} d\theta \\
&= \int_0^{2\pi}\left\{\int_o^{2\pi} \left|\left(R^n P(e^{i\theta}/R) - r^n P(e^{i\theta}/r)\right)e^{i\gamma} + \left(P(Re^{i\theta}) - P(re^{i\theta})\right)\right|^p d\theta\right\} d\gamma \\
&\geq 2\pi\left(R^n - r^n\right)^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta\,. \qquad (3.3)
\end{aligned}$$

Now it can be easily verified that for every real number γ and $s \geq q \geq 1$,

$$\left|s + e^{i\alpha}\right| \geq \left|q + e^{i\alpha}\right|.$$

If $F(\theta) \neq 0$, we take $s = |G(\theta)/F(\theta)|$ and $q = k^\mu \phi(R,r,\mu,k)$, then by (1.8) and (3.1), $s \geq q \geq 1$, we get using (3.2), This implies for each $p > 0$,

$$\begin{aligned}\int_0^{2\pi} \left|F(\theta) + e^{i\gamma}G(\theta)\right|^p d\gamma &= |F(\theta)|^p \int_0^{2\pi} \left|1 + e^{i\gamma}\frac{G(\theta)}{F(\theta)}\right|^p d\gamma \\ &= |F(\theta)|^p \int_0^{2\pi} \left|e^{i\gamma} + \frac{G(\theta)}{F(\theta)}\right|^p d\gamma \\ &= |F(\theta)|^p \int_0^{2\pi} \left|e^{i\gamma} + \left|\frac{G(\theta)}{F(\theta)}\right|\right|^p d\gamma \\ &\geq |F(\theta)|^p \int_0^{2\pi} \left|k^\mu\phi(R,r,\mu,k) + e^{i\gamma}\right|^p d\gamma \\ = \Bigg| \left|P(Re^{i\theta}) - P(re^{i\theta})\right| &+ \frac{t(R^n - r^n)}{1 + k^\mu\phi(R,r,\mu,k)} m(P,k)\Bigg|^p \\ &\times \int_0^{2\pi} \left|k^\mu\phi(R,r,\mu,k) + e^{i\gamma}\right|^p d\gamma\,. \end{aligned} \tag{3.4}$$

If $F(\theta) = 0$, then (3.4) is trivially true. Using this in (3.3), we conclude for each $R > r \geq 1$ and $p > 0$,

$$\int_0^{2\pi} \left| \left|P(Re^{i\theta}) - P(re^{i\theta})\right| + \frac{t(R^n - r^n)}{1 + k^\mu\phi(R,r,\mu,k)} m(P,k)\right|^p d\theta \times \int_0^{2\pi} \left|k^\mu\phi(R,r,\mu,k) + e^{i\gamma}\right|^p d\gamma \leq 2\pi\big(R^n - r^n\big)^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

This gives for every real or complex number δ with $|\delta| \leq 1$, $R > r \geq 1$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} \left| \big(P(Re^{i\theta}) - P(re^{i\theta})\big) + \delta\frac{t(R^n - r^n)}{1 + k^\mu\phi(R,r,\mu,k)} m(P,k)\right|^p d\theta\right\}^{\frac{1}{p}} \leq \frac{R^n - r^n}{\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|k^\mu\phi(R,r,\mu,k) + e^{i\gamma}\right|^p d\gamma\right\}^{\frac{1}{p}}} \left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}.$$

This completes the proof of Theorem 1. □

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