

**SOME INEQUALITIES
FOR THE DERIVATIVE OF A POLYNOMIAL**

Nisar Ahmad Rather¹, Sajad Hussian Ahangar², Mushtaq Ahmad Shah^{3 §}

^{1,2,3}P.G. Department of Mathematics

University of Kashmir

Srinagar, Hazratbal, 190006, INDIA

Abstract: Let $P_n(z)$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^n a_j z^j$ of degree n . According to a well known inequality of S. Bernstein, if $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$

In this paper, we establish some generalizations and refinements of the above inequality and some other well known inequalities concerning the polynomials and their derivatives.

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1. Introduction and Statement of Results

Let $P_n(z)$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^n a_j z^j$ of degree n . According to a well known result due to S. Bernstein (see [12]), if $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \quad (1)$$

Also a simple deduction from the maximum modulus principle yields that, if $P \in P_n$, then

$$\max_{|z|=R>1} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \quad (2)$$

Both the inequalities (1) and (2) are sharp and the equality in (1) and (2) holds if and only if $P(z)$ has all its zeros at the origin. It was shown by Frappier, Rahman and Ruscheweyh [5, Theorem 8] that if $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{1 \leq k \leq n} |P(e^{ik\pi n})|. \quad (3)$$

Clearly (3) represents a refinement of (1), since the maximum of $|P(z)|$ on $|z| = 1$ may be larger than the maximum of $|P(z)|$ taken over $(2n)^{th}$ roots of unity, as is shown by the simple example $P(z) = z^n + ia, a > 0$.

A. Aziz [1] showed that the bound in (3) can be considerably improved. In fact proved that, if $P \in P_n$, then for every given real α ,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} (M_\alpha + M_{\alpha+\pi}), \quad (4)$$

where

$$M_\alpha = \max_{1 \leq k \leq n} |P(e^{i(\alpha+2k\pi)n})| \quad (5)$$

and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha + \pi$. The result is best possible and equality in (4) holds for $P(z) = z^n + re^{i\alpha}, -1 \leq r \leq 1$.

Clearly inequality (4) is an interesting refinement of inequality (3), and hence of the Bernstein inequality (1).

If we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$, then the inequality (1) can be sharpened. In fact, P. Erdos conjectured and later P.D. Lax [7] (see also [2]), verified that if $P(z) \neq 0$ for $|z| < 1$, then (1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \quad (6)$$

In this connection, A. Aziz [1], improved the bound in inequality (4) by showing that if $P \in P_n$ and $P(z)$ has no zero in $|z| < 1$, then for every real α ,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \quad (7)$$

where M_α is defined by (5). The result is best possible and equality in (7) holds for $P(z) = z^n + e^{i\alpha}$.

A. Aziz [1] also proved that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real α and $R > 1$,

$$\max_{|z|=1} |P(Rz) - P(z)| \leq \frac{R^n - 1}{2} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}. \quad (8)$$

In this paper, we consider the class $P_{n,\mu}(z)$ of all complex polynomials $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n$ of degree n and obtain some generalizations of the inequalities (7) and (8). In this direction, we first prove the following generalization of the inequality (7).

Theorem 1.1. *If $P \in P_{n,\mu}$, having no zero in $|z| < k, k \geq 1$, then for every real α ,*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \quad (9)$$

where M_α is defined by (5).

As an application of Theorem 1.1, we next prove the following result, which is a corresponding generalization of the inequality (8).

Theorem 1.2. *If $P \in P_{n,\mu}$, having no zero in $|z| < k, k \geq 1$, then for every real α and $R > 1$,*

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \quad (10)$$

where M_α is defined by (5).

Instead of proving Theorems 1.1 and 1.2, we prove the following results, which refine as well as generalize inequalities (7) and (8), respectively.

Theorem 1.3. *If $P \in P_{n,\mu}$, having no zero in $|z| < k, k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every real α ,*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \quad (11)$$

where M_α is defined by (5).

Remark 1.4. For $\mu = 1$, Theorem 1.3 gives a generalization and a refinement of the inequality (7).

As an application of Theorem 1.3, we next prove the following result, which is a corresponding generalization and a refinement of the inequality (8).

Theorem 1.5. *If $P \in P_{n,\mu}$, having no zero in $|z| < k, k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every real α and $R > 1$,*

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \quad (12)$$

where M_α is defined by (5).

By involving some coefficients of the polynomial $P_{n,\mu}(z)$, we also present the following refinement of Theorem 1.1.

Theorem 1.6. If $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$ is a polynomial of degree n , having no zero in $|z| < k, k \geq 1$, then for every real α ,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2 \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1 \right)^2 \right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \quad (13)$$

where M_α is defined by (5).

As an application of Theorem 1.6, we next prove the following result.

Theorem 1.7. If $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$ is a polynomial of degree n , having no zero in $|z| < k, k \geq 1$, then for every real α and $R > 1$,

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2 \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1 \right)^2 \right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}. \quad (14)$$

Finally, we prove the following refinement of the inequality (13).

Theorem 1.8. If $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$ is a polynomial of degree n , having no zero in $|z| < k, k \geq 1$, and $m = \min_{|z|=k} |P(z)|$, then for every real α ,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2 \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1 \right)^2 \right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \quad (15)$$

where M_α is defined by (5).

As an application of Theorem 1.8, one can easily obtain the following result.

Theorem 1.9. If $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$ is a polynomial of degree n , having no zero in $|z| < k, k \geq 1$, and $m = \min_{|z|=k} |P(z)|$, then for every real α and $R > 1$,

$$\begin{aligned} & |P(Rz) - P(z)| \\ & \leq \frac{R^n - 1}{\sqrt{2 \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1 \right)^2 \right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}. \quad (16) \end{aligned}$$

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first lemma is due to A. Aziz [1].

Lemma 2.1. *If $P(z)$ is a polynomial of degree n , then for $|z| = 1$ and for every real α ,*

$$|P'(z)|^2 + |nP(z) - zP'(z)|^2 \leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2), \quad (17)$$

where M_α is defined by (5).

Lemma 2.2. *If $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$ is a polynomial of degree at most n and $P(z) \neq 0$ for $|z| < k, k \geq 1$, then for $|z| = 1$,*

$$k^\mu |P'(z)| \leq |nP(z) - zP'(z)| - nm,$$

where $m = \min_{|z|=k} |P(z)|$.

Lemma 2.2 is a special case of a lemma due to A. Aziz and N.A. Rather [3, Lemma 5].

We also need the following lemma which is implicit in [9, Lemma 1].

Lemma 2.3. *If $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$ is a polynomial of degree at most n and $P(z) \neq 0$ for $|z| < k, k \geq 1$, then for $|z| = 1$,*

$$k^{\mu+1} \frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1}{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu+1}} |P'(z)| \leq |nP(z) - zP'(z)|.$$

Lemma 2.4. *If $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$ is a polynomial of degree at most n and $P(z) \neq 0$ for $|z| < k, k \geq 1$, and $m = \min_{|z|=k} |P(z)|$, then for $|z| = 1$,*

$$k^{\mu+1} \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu-1} + 1}{1 + \frac{\mu}{n} \frac{|a_\mu|}{|a_0|-m} k^{\mu+1}} |P'(z)| \leq |nP(z) - zP'(z)| - nm.$$

This lemma is implicit in [10, Lemma 10].

3. Proof of Theorems

Proof of Theorem 1.3. By hypothesis, $P(z)$ does not vanish in $|z| < k, k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, therefore by Lemma 2.2, we have

$$(k^\mu |P'(z)| + nm)^2 \leq |nP(z) - zP'(z)|^2 \quad \text{for } |z| = 1.$$

This gives with the help of Lemma 2.1 for $|z| = 1$,

$$\begin{aligned} |P'(z)|^2 + (k^\mu |P'(z)| + nm)^2 &\leq |P'(z)|^2 + |nP(z) - zP'(z)|^2 \\ &\leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2). \end{aligned}$$

Since

$$(k^\mu |P'(z)| + nm)^2 = k^{2\mu} |P'(z)|^2 + n^2 m^2 + 2nmk^\mu |P'(z)| \geq k^{2\mu} |P'(z)|^2 + n^2 m^2,$$

it follows that

$$(1 + k^{2\mu}) |P'(z)|^2 + n^2 m^2 \leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2),$$

which implies for $|z| = 1$

$$|P'(z)| \leq \frac{n}{\sqrt{2(1 + k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}$$

and hence

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2(1 + k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}.$$

This completes the proof of Theorem 1.3.

Proof of Theorem 1.5. Applying (2) to the polynomial $P'(z)$ which is of degree $n - 1$ and using Theorem 1.1, we obtain for $t \geq 1$ and $0 \leq \theta < 2\pi$,

$$|P'(te^{i\theta})| \leq t^{n-1} \max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2(1 + k^{2\mu})}} t^{n-1} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}.$$

Hence for each $\theta, 0 \leq \theta < 2\pi$ and $R > 1$, we have

$$\begin{aligned} |P(Re^{i\theta}) - P(e^{i\theta})| &= \left| \int_1^R e^{i\theta} P'(te^{i\theta}) dt \right| \\ &\leq \int_1^R |P'(te^{i\theta})| dt \\ &\leq \frac{1}{\sqrt{2(1 + k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2} \int_1^R nt^{n-1} dt. \\ &= \frac{1}{\sqrt{2(1 + k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2} (R^n - 1). \end{aligned}$$

This implies for $|z| = 1$ and $R > 1$,

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2(1 + k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}.$$

This proves Theorem 1.5.

Proof of Theorem 1.6. By hypothesis $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$ is a polynomial of degree at most n and $P(z) \neq 0$ for $|z| < k, k \geq 1$, therefore by Lemma 2.3, it follows that

$$k^{\mu+1} \frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1}{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu+1}} |P'(z)| \leq |nP(z) - zP'(z)|, \text{ for } |z| = 1,$$

which implies

$$k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1}{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu+1}} \right)^2 |P'(z)|^2 \leq |nP(z) - zP'(z)|^2, \text{ for } |z| = 1.$$

This gives with the help of Lemma 2.1

$$\begin{aligned} \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1}{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu+1}} \right)^2 \right\} |P'(z)|^2 &\leq |P'(z)|^2 + |nP(z) - zP'(z)|^2 \\ &\leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2), \end{aligned}$$

equivalently

$$|P'(z)|^2 \leq \frac{n^2}{2 \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1}{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu+1}} \right)^2 \right\}} (M_\alpha^2 + M_{\alpha+\pi}^2).$$

This gives

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2 \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1}{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu+1}} \right)^2 \right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}.$$

This completes the proof of Theorem 1.6.

Proof of Theorem 1.7. Applying (2) to the polynomial $P'(z)$ which is of degree $n-1$ and using Theorem 1.3, we obtain for $t \geq 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned} \left| P'(te^{i\theta}) \right| &\leq t^{n-1} \max_{|z|=1} |P'(z)| \\ &\leq \frac{nt^{n-1}}{\sqrt{2 \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1 \right)^2 \right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}. \end{aligned}$$

Hence for each $\theta, 0 \leq \theta < 2\pi$ and $R > 1$, we have

$$\begin{aligned} &\left| P(Re^{i\theta}) - P(e^{i\theta}) \right| \\ &= \left| \int_1^R e^{i\theta} P'(te^{i\theta}) dt \right| \\ &\leq \int_1^R \left| P'(te^{i\theta}) \right| dt \\ &\leq \frac{1}{\sqrt{2 \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1 \right)^2 \right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2} \int_1^R nt^{n-1} dt \\ &= \frac{1}{\sqrt{2 \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1 \right)^2 \right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2} (R^n - 1). \end{aligned}$$

This implies for $|z| = 1$ and $R > 1$,

$$\left| P(Rz) - P(z) \right| \leq \frac{R^n - 1}{\sqrt{2 \left\{ 1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1 \right)^2 \right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}.$$

Hence the proof of Theorem 1.7 is complete.

The proofs of Theorems 1.8 and 1.9 follow on the same lines as that of Theorems 1.3 and 1.5, except that instead of Lemma 2.3, we use Lemma 2.4.

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