A FLUID MODEL DRIVEN BY AN M/M/1 QUEUE
WITH CATASTROPHE AND RESTORATION TIME

T. Vijayalakshmi[1], V. Thangaraj[2] [§]

[1,2]Ramanujan Institute for Advanced Study in Mathematics
University of Madras
Chepauk, Chennai, 600 005, INDIA

Abstract: In this paper, we discuss a fluid queue driven by an M/M/1 queue with catastrophe and restoration. Using the continued fraction technique, we derive the Laplace transform of the transient state probability distribution of the buffer content, transient solution for this fluid queue and the corresponding steady state results explicitly. Also we analyze the queue when there are no catastrophes and no restoration.

AMS Subject Classification: 60K25, 11Y65
Key Words: fluid queue, M/M/1 with catastrophe, restoration time, three-term recurrence relation, Laplace transform, inverse Laplace transform

1. Introduction

Markov process modulated fluid models play a significant role in the ATM networks (Elvalid and Mitra [4], Anick et. al. [2], Simonian and Virtomo [12]). In addition fluid models driven by finite state space Markov processes that modulate the input rate in the fluid buffer have been analyzed by many authors (Anick et al. [2], Coffman et al. [3], Gaver et al. [5], Mitra [9], Low and Varaiya [8]). Virtoma and Norros [14] have proposed a spectral-decomposition method to analyze a fluid model driven by an M/M/1 queue. Adan and Resing [1] have analyzed and expressed the generalized eigen values explicitly using the

---

[§]Correspondence author

Chebyshev polynomials of the second kind. Van Doorn and Scheinhardt [13] have obtained explicit expressions for the stationary distribution of the buffer content for fluid queues driven by an M/M/1 queue with constant arrival service rates. A continued fraction method to analyse fluid queues has been proposed by Parthasarathy, Lenin and Vijayashree [10]. Transmission Control Protocol (TCP) is able to guarantee that each data packet transmitted from a server (computer) and leave it momentarily inactivated until the new arrival occurs, such infected cells may be modeled by catastrophes. Jain and Rakesh Kumar [6] have obtained transient solution of the model with correlated arrival queueing with variable capacity and catastrophes for the cell traffic generated by New Broadband Communication Networks in presence of viruses and noise bursts. The catastrophe makes the queue instantly empty whenever the system is not empty but the system takes its own time to be ready to accept new customers, this repair time at the service facility is referred to as the restoration time and solved a correlated queueing system with catastrophic-cum-restorative effects (Jain and Rakesh Kumar [7]). The applications of the model can be found in a variety of areas, especially in new broadband communication networks. This situation forces us to construct a fluid model.

However, as there is no result on a fluid model driven by a M/M/1 queue with catastrophe, and restoration time, we study such a fluid model. We have derived the Laplace transform of the transient state distribution of the buffer content through continued fractions. We have also obtained the transient solution for M/M/1 fluid queue and explicitly obtained the corresponding steady state results. We have further analyzed when there is no catastrophe and no restoration.

The rest of the paper is organized as follows. The mathematical description of our model is in Section 2 and the equations governing the model are given in Section 3. The transient solution has been derived in Section 4 and the corresponding steady state results have been obtained and analyzed when there are no catastrophes and restoration, in Section 5.

2. Mathematical Description of the Model

Consider a fluid model with an infinite buffer driven by an M/M/1 queue with catastrophe and restoration time. The background process $\{X(t), \ t \geq 0\}$ takes values in $S = \{0, 1, 2, \ldots\}$, where $X(t)$ denotes the number of customers in the system at time t. Let λ and μ denote the arrival and service rates respectively. The arrivals follow a Poisson process with rate λ and the service times are ex-

ponentially distributed with a mean μ^{-1}. When the system is not empty (the server is busy in service time), catastrophe occurs at the service facility according to Poisson process with rate ξ. Occurrence of a catastrophe to the busy server annihilates the entire system and the server is subject to catastrophic failure. In practice, any system suffering from catastrophe must take some time for its restoration. The repair times of failed server are i.i.d according to an exponential distribution with mean η^{-1}. After the completion of restoration period, the server immediately return to its working position for service when a new customer arrives. Let $R(t)$ be the probability of restoration time when the service facility under restoration (independent of buffer content). This situation prompts to construct a stochastic fluid queueing model.

3. Equations Governing the System

A fluid commodity which we refer to as credit *accumulates* in an infinite fluid buffer at a constant rate $r > 0$ during the *busy period of the server*. The credit buffer *depletes* the fluid during the *idle periods of the server* at a constant rate $r_0 < 0$ as long as the buffer is nonempty. We denote by $C(t)$, the content of buffer at time t. The 2-dimensional process $\{X(t), C(t), \ t \geq 0\}$ constitutes a Markov process. Define

$$F_j(t, x) = P\{X(t) = j, \ C(t) \leq x\}, \quad j \in S, t, x \geq 0.$$

Theorem 3.1. *For the above fluid model with catastrophe and restoration as described in Section 2, the sequence $\{F_0(t,x), F_j(t,x), R(t)\}$ satisfies the following system of differential equations:*

$$\frac{\partial}{\partial t}F_0(t,x) + r_0\frac{\partial}{\partial x}F_0(t,x) \ = \ -\lambda F_0(t,x) + \mu F_1(t,x) + \eta R(t) \qquad (3.1)$$

$$\frac{\partial}{\partial t}F_j(t,x) + r\frac{\partial}{\partial x}F_j(t,x) \ = \ -(\lambda + \mu + \xi)F_j(t,x) + \lambda F_{j-1}(t,x)$$
$$+ \mu F_{j+1}(t,x), j = 1,2,3,\ldots, t, x \geq 0 \ (3.2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}R(t) \ = \ -\eta R(t) + \xi(1 - R(t) - q_0(t)) \qquad (3.3)$$

subject to the initial conditions

$$F_0(0,x) = 1, F_j(0,x) = 0, j = 1,2,3,\ldots \qquad (3.4)$$

and boundary conditions

$$F_j(t,0) = q_j(t), j = 0, 1, 2, \ldots \qquad (3.5)$$

Proof. Using the above assumptions in Section 2, by imitating the simple probability arguments as in Anick, Mitra and Sondhi [2], the system of forward differential equations satisfied by the Markov process $\{X(t), C(t), \ t \geq 0\}$ and the restoration probability $R(t)$ are derived. Because of their simplicity, the details of the proof are omitted. $\qquad\square$

Remark 3.1. Here $q_j(t)$ represents the probability that at time t the buffer is empty and the state of the background Markov process is j. The content buffer decreases and thereby becomes empty only when the net input rate of the fluid into the buffer is positive. Therefore, when the buffer becomes empty at any time t the background process should necessarily be in state zero corresponding to which the effective input rate is $r_0 < 0$. Hence we have $q_j(t) = 0$ for $j = 1, 2, 3, \ldots$ as $r_j = r > 0$ for $j = 1, 2, 3, \ldots$.

4. Transient Solution

In this section, we obtain explicitly the transient solution for the fluid model with catastrophe-cum-restoration by the method of continued fractions. Simplicity and elegance are the strong points of this method.

Theorem 4.1. *Let $F_j^{**}(s, \omega), R^*(s)$ be the Laplace transforms of $F_j(t, x)$, $j = 0, 1, 2, \ldots, R(t)$, respectively. Then*

$$F_0^{**}(s, \omega) = \left[q_0^*(s) + \frac{1 + \eta R^*(s)}{r_0 \omega} \right] \sum_{k=0}^{\infty} \left(\frac{r}{2r_0} \right)^k \frac{(\theta - \sqrt{\theta^2 - \alpha^2})^k}{\left(\omega + \dfrac{s + \lambda}{r_0} \right)^{k+1}}$$

$$\text{for } \left| \frac{\lambda \mu}{r_0 \omega + s + \lambda} \right| < 1, \quad (4.1)$$

and for $j = 1, 2, 3, \ldots$,

$$F_j^{**}(s, \omega) = \left(\frac{r}{2\mu} \right)^j \left[q_0^*(s) + \frac{1 + \eta R^*(s)}{r_0 \omega} \right] \sum_{k=0}^{\infty} \frac{(\theta - \sqrt{\theta^2 - \alpha^2})^{j+k}}{\left(\omega + \dfrac{s + \lambda}{r_0} \right)^{k+1}} \quad (4.2)$$

$$R^*(s) \quad = \quad \frac{\xi}{(s+\eta+\xi)}\left(\frac{1}{s} - q_0^*(s)\right). \tag{4.3}$$

Proof. Taking Laplace transforms of (3.1)-(3.3) and using (3.4), (3.5) with respect to t and x, we obtain

$$(s+r_0\omega+\lambda)F_0^{**}(s,w) - \mu F_1^{**}(s,w) = \frac{\eta}{\omega}R^*(s) + r_0 q_0^*(s) + \frac{1}{\omega} \tag{4.4}$$

$$-\lambda F_{j-1}^{**}(s,w) + (s+\lambda+\mu+\xi+r\omega)F_j^{**}(s,w) - \mu F_{j+1}^{**}(s,w) = 0, j = 1,2,3,\ldots \tag{4.5}$$

and

$$(s+\eta+\xi)R^*(s) = \xi\left(\frac{1}{s} - q_0^*(s)\right). \tag{4.6}$$

Equations (4.4) and (4.5) can be conveniently written respectively in the form of continued fractions as follows:

$$F_0^{**}(s,\omega) = \frac{\left[\frac{1}{\omega} + \frac{\eta R^*(s)}{\omega} + r_0 q_0^*(s)\right]}{(s+r_0\omega+\lambda) - \mu\dfrac{F_1^{**}(s,\omega)}{F_0^{**}(s,\omega)}} \tag{4.7}$$

and

$$(s+\lambda+\mu+\xi+r\omega) = \lambda\frac{F_{j-1}^{**}(s,\omega)}{F_j^{**}(s,\omega)} + \mu\frac{F_{j+1}^{**}(s,\omega)}{F_j^{**}(s,\omega)}, j = 1,2,3,\ldots \tag{4.8}$$

The above equation leads to

$$\frac{F_j^{**}(s,\omega)}{F_{j-1}^{**}(s,\omega)} = \frac{\lambda}{(s+\lambda+\mu+\xi+r\omega) - \mu\dfrac{F_{j+1}^{**}(s,\omega)}{F_j^{**}(s,\omega)}}, j = 1,2,3,\ldots \tag{4.9}$$

For the purpose of algebraic simplification, define

$$\begin{aligned} f(s,\omega) \quad &= \quad \frac{1}{(s+\lambda+\mu+\xi+r\omega) - \frac{\lambda\mu}{s+\lambda+\mu+\xi+r\omega-}\cdots} \\ &= \quad \frac{1}{(s+\lambda+\mu+\xi+r\omega) - \lambda\mu f(s,\omega)} \end{aligned}$$

i.e., $\quad \lambda\mu f^2(s,\omega) - (s+\lambda+\mu+r\omega+\xi)f(s,\omega) + 1 = 0.$

Solving the above quadratic equation, we obtain by taking the root less than 1,

$$f(s,\omega) = \frac{(s+\lambda+\mu+r\omega+\xi) - \sqrt{(s+\lambda+\mu+r\omega+\xi) - 4\lambda\mu}}{2\lambda\mu}.$$

Using the above definition, we have

$$\frac{F_j^{**}(s,\omega)}{F_{j-1}^{**}(s,\omega)} = \lambda f(s,\omega), \quad j = 1,2,3,\ldots. \qquad (4.10)$$

Hence

$$F_0^{**}(s,\omega) = \frac{\left[\frac{1}{\omega} + \dfrac{\eta R^*(s)}{\omega} + r_0 q_0^*(s)\right]}{(s+r_0\omega+\lambda) - \lambda\mu f(s,\omega)}$$

$$= \frac{\dfrac{1}{\omega} + \dfrac{\eta R^*(s)}{\omega} + r_0 q_0^*(s)}{s+r_0\omega+\lambda - \dfrac{(s+\lambda+\mu+r\omega+\xi) - \sqrt{(s+\lambda+\mu+r\omega+\xi)^2 - 4\lambda\mu}}{2}}.$$

$$(4.11)$$

Let $\omega + \dfrac{s+\lambda+\mu+\xi}{r} = \theta, \quad \dfrac{2\sqrt{\lambda\mu}}{r} = \alpha.$ Then we write (4.11) as

$$F_0^{**}(s,\omega) = \frac{\dfrac{1}{\omega} + \dfrac{\eta R^*(s)}{\omega} + r_0 q_0^*(s)}{\left[r_0\left(\dfrac{s+\lambda}{r_0} + \omega\right) - \dfrac{r}{2}(\theta - \sqrt{\theta^2 - \alpha^2})\right]}$$

and

$$F_j^{**}(s,\omega) = (\lambda f(s,\omega))^j F_0^{**}(s,\omega), j = 1,2,3,\ldots.$$

By simple mathematical calculations, we complete the proof of the theorem. $\quad\square$

We present below the transient state joint probability distribution of our fluid model.

Theorem 4.2.   *For every $t \geq 0$ and $x \in [0, rt)$, the transient state joint probability distributions of a fluid queue driven by an M/M/1 queue with catastrophes and restoration time are:*

$$F_0(t, x) = e^{\frac{-\lambda x}{r_0}} q_0(t - \frac{x}{r_0}) + \frac{\xi\eta}{\lambda(\eta+\xi)}(1 + e^{\frac{-\lambda x}{r_0}}) + \frac{2\xi\eta e^{-\lambda t}}{\lambda(\lambda - (\eta+\xi))}$$

$$+ \frac{\xi\eta}{(\eta+\xi)(\eta+\xi-\lambda)}[e^{-(\eta+\xi)t} + e^{-(\eta+\xi)(t-\frac{x}{r_0})-\frac{\lambda x}{r_0}}]$$

$$+ e^{-\lambda t} - e^{-\frac{\lambda x}{r_0}}e^{-\lambda(t-\frac{x}{r_0})} + \sum_1^\infty \left(\frac{r}{2r_0}\right)^k \int_0^x e^{-\frac{\lambda+\mu+\xi}{r}(x-y)} \frac{\alpha^k k I_k(\alpha(x-y))}{(x-y)}$$

$$\times \left(\frac{y^k e^{-\lambda y/r_0}}{k!}H\left(t - \frac{x-y}{r} - \frac{y}{r_0}\right)q_0\left(t - \frac{x-y}{r} - \frac{y}{r_0}\right)\right.$$

$$+ \frac{r_0^k e^{-\lambda(t-\frac{x-y}{r})}\left(t-\frac{x-y}{r}\right)^k}{k!} + \eta\xi\frac{r_0^k}{k!}\int_0^t \left[e^{-\lambda\left(t-u-\frac{x-y}{r}\right)}\left(t-u-\frac{x-y}{r}\right)^k\right.$$

$$\left.\left.\times \int_0^u (1 - q_0(v))e^{-(\eta+\xi)(u-v)}dv\right]du\right)dy,$$

$$\tag{4.12}$$

$$F_j(t, x) = \left(\frac{r}{2\mu}\right)^j \sum_0^\infty \left(\frac{r}{2r_0}\right)^k \int_0^x e^{-\frac{\lambda+\mu+\xi}{r}(x-y)} \frac{(j+k)\alpha^{j+k}I_{j+k}(\alpha(x-y))}{k!(x-y)}$$

$$\times \left(y^k e^{-\lambda y/r_0} H\left(t - \frac{x-y}{r} - \frac{y}{r_0}\right)q_0\left(t - \frac{x-y}{r} - \frac{y}{r_0}\right)\right.$$

$$+ r_0^k e^{-\lambda(t-\frac{x-y}{r})}\left(t - \frac{x-y}{r}\right)^k$$

$$+ \eta\xi\frac{r_0^k}{k!}\int_0^t \left[e^{-\lambda\left(t-u-\left(\frac{x-y}{r}\right)\right)}\left(t - u - \left(\frac{x-y}{r}\right)\right)^k\right.$$

$$\left.\left.\times \int_0^u (1 - q_0(v))e^{-(\eta+\xi)(u-v)}dv\right]du\right)dy \quad j = 1, 2, 3, \ldots, \tag{4.13}$$

$$R(t) = \xi \int_0^t [1 - q_0(v)]e^{-(\xi+\eta)(t-v)}dv, \tag{4.14}$$

where $H(.)$ is the Heaviside function and $I_k(.)$ is the modified Bessel function of the first kind.

Proof. To obtain the transient solutions of the system, we invert $F_j^{**}(s, \omega)$ for $j = 1, 2, 3, \ldots$ with respect to s and ω. From (4.1),

$F_0^{**}(s, \omega)$

$$
= \left[q_0^*(s) + \frac{1 + \eta R^*(s)}{r_0 \omega} \right] \left[\frac{1}{(\omega + \frac{s + \lambda}{r_0})} + \sum_{k=1}^{\infty} \left(\frac{r}{2r_0} \right)^k \frac{(\theta - \sqrt{\theta^2 - \alpha^2})^k}{\left(\omega + \frac{s + \lambda}{r_0} \right)^{k+1}} \right]
$$

$$
= \frac{q_0^*(s)}{\left(\omega + \frac{s + \lambda}{r_0} \right)} + \frac{1 + \eta R^*(s)}{r_0 \omega \left(\omega + \frac{s + \lambda}{r_0} \right)} + q_0^*(s) \sum_{k=1}^{\infty} \left(\frac{r}{2r_0} \right)^k \frac{(\theta - \sqrt{\theta^2 - \alpha^2})^k}{\left(\omega + \frac{s + \lambda}{r_0} \right)^{k+1}}
$$

$$
+ \frac{(1 + \eta R^*(s))}{r_0 \omega} \sum_{k=1}^{\infty} \left(\frac{r}{2r_0} \right)^k \frac{(\theta - \sqrt{\theta^2 - \alpha^2})^k}{\left(\omega + \frac{s + \lambda}{r_0} \right)^{k+1}}. \tag{4.15}
$$

On inverting (4.15) with respect to ω, we get

$F_0^*(s, x)$

$$
= q_0^*(s) e^{-\frac{\lambda x}{r_0}} e^{-\frac{sx}{r_0}} + \frac{1 + \eta R^*(s)}{s + \lambda} - \frac{1 + \eta R^*(s)}{s + \lambda} e^{-\frac{\lambda x}{r_0}} e^{-\frac{sx}{r_0}}
$$

$$
+ \sum_{k=1}^{\infty} \left(\frac{r}{2r_0} \right)^k \int_0^x e^{-\frac{(\lambda + \mu + \xi)}{r}(x-y)} \frac{\alpha^k k I_{k(\alpha(x-y))}}{x - y} \left[\frac{y^k e^{-\frac{\lambda y}{r_0}}}{k!} q_0^*(s) e^{-\frac{s(x-y)}{r}} e^{-\frac{sy}{r_0}} \right.
$$

$$
\left. + \frac{1 + \eta R^*(s)}{r_0} e^{-\frac{s(x-y)}{r}} h_k(s, y) \right] dy, \tag{4.16}
$$

where

$$
h_k(s, y) = \int_0^y \frac{u^k e^{-\frac{(s + \lambda + \xi)u}{r_0}}}{k!} du. \tag{4.17}
$$

Multiplying both sides by $e^{\frac{sx}{r_0}}$ we get

$$e^{\frac{sx}{r_0}}F_0^*(s,x) = q_0^*(s)e^{-\frac{\lambda x}{r_0}} + \frac{1+\eta R^*(s)}{s+\lambda}e^{\frac{sx}{r_0}} - \frac{1+\eta R^*(s)}{s+\lambda}e^{-\frac{\lambda x}{r_0}}$$

$$+ \sum_{k=1}^{\infty}\left(\frac{r}{2r_0}\right)^k \int_0^x e^{-\frac{(\lambda+\mu+\xi)}{r}(x-y)}\frac{\alpha^k k I_{k(\alpha(x-y))}}{x-y}\left[\frac{y^k e^{-(\frac{\lambda y}{r_0})}}{k!}q_0^*(s)e^{-s(\frac{1}{r}-\frac{1}{r_0})(x-y)}\right.$$

$$\left. + \frac{1+\eta R^*(s)}{r_0}e^{-\frac{s(x-y)}{r}}h_k(s,y)e^{\frac{sx}{r_0}}\right]dy. \tag{4.18}$$

Inverting both sides of (4.18) with respect to s variable, we get

$$H(t+\frac{x}{r_0})F_0(t+\frac{x}{r_0},x)$$

$$= e^{\frac{-\lambda x}{r_0}}q_0(t) + H(t+\frac{x}{r_0})e^{-\lambda(t+\frac{x}{r_0})} + \frac{\xi\eta}{\lambda(\eta+\xi)}H(t+\frac{x}{r_0}) + \frac{\xi\eta}{\lambda(\lambda-(\eta+\xi))}e^{-\lambda(t+\frac{x}{r_0})}$$

$$+ \frac{\xi\eta e^{-(\eta+\xi)(t+\frac{x}{r_0})}H(t+\frac{x}{r_0})}{(\eta+\xi)((\eta+\xi)-\lambda)} - e^{\frac{-\lambda x}{r_0}}e^{-\lambda t} + \frac{\xi\eta}{\lambda(\eta+\xi)}e^{\frac{-\lambda x}{r_0}} + \frac{\xi\eta}{\lambda(\lambda-(\eta+\xi))}e^{-\lambda(t+\frac{x}{r_0})}$$

$$+ \frac{\xi\eta e^{-(\eta+\xi)t}e^{\frac{-\lambda x}{r_0}}}{(\eta+\xi)(\eta+\xi-\lambda)}$$

$$+ \sum_1^{\infty}(\frac{r}{2r_0})^k \int_0^x e^{-\frac{\lambda+\mu+\xi(x-y)}{r}}\frac{\alpha^k k I_{k(\alpha(x-y))}}{(x-y)}$$

$$\left(\frac{y^k e^{-\lambda y/r_0}}{k!}H(t-\frac{x-y}{r}+\frac{x-y}{r_0})q_0(t-\frac{x-y}{r}+\frac{x-y}{r_0})\right.$$

$$+ \frac{r_0^k e^{-\lambda(t-\frac{x-y}{r}+\frac{x}{r_0})}(t-\frac{x-y}{r}+\frac{x}{r_0})^k}{k!}$$

$$+ \eta\xi\frac{r_0^k}{k!}\int_0^t\left[e^{-\lambda(t-u-(\frac{x-y}{r}-\frac{x}{r_0}))}\left(t-u-\left(\frac{x-y}{r}-\frac{x}{r_0}\right)\right)^k\right.$$

$$\left.\left. \times \int_0^u (1-q_0(v))e^{-(\eta+\xi)(u-v)}dv\right]du\right)dy. \tag{4.19}$$

After simplification, we get the required result for $F_0(t,x)$.

Now, consider

$$F_j^{**}(s,\omega) = (\lambda f(s,\omega))^j F_0^{**}(s,\omega), \quad \text{for } j=1,2,3,\ldots$$

$$= \left(\frac{r}{2\mu}\right)^j\left[q_0^*(s)+\frac{1+\eta R^*(s)}{r_0\omega}\right]\sum_{k=0}^{\infty}\frac{(\theta-\sqrt{\theta^2-\alpha^2})^{j+k}}{\left(\omega+\frac{s+\lambda}{r_0}\right)^{k+1}}. \tag{4.20}$$

By similar arguments as before, inverting (4.20) with respect to w, we get

$$
\begin{aligned}
F_j^*(s,x) &= \left(\frac{r}{2\mu}\right)^j \sum_0^\infty \left(\frac{r}{2r_0}\right)^k \int_0^x e^{-\frac{\lambda+\mu+\xi}{r}(x-y)} \frac{(j+k)\alpha^{j+k} I_{j+k}(\alpha(x-y))}{k!(x-y)} \\
&\quad \times \left[\frac{y^k e^{-\lambda y/r_0}}{k!} q_0^*(s) e^{-s(\frac{x-y}{r}-\frac{y}{r_0})} + \frac{[1+\eta R^*(s)]}{r_0} e^{-s(\frac{x-y}{r})h_k(s,y)}\right] dy.
\end{aligned}
$$

$$(4.21)$$

Now inverting (4.21) and (4.3) with respect to s by similar arguments as before, we get the required other two results.   □

Particular Case 1: When the restoration times R are zero (i.e. $\eta \to \infty$), we obtain

$$
\begin{aligned}
F_0(t,x) &= e^{\frac{-\lambda x}{r_0}} q_0(t - \frac{x}{r_0}) + \frac{\xi(1+e^{-\lambda \frac{x}{r_0}})}{\lambda} + e^{-\lambda t} - e^{-\frac{\lambda x}{r_0}} e^{-\lambda(t-x/r_0)} \\
&\quad + \frac{2\xi e^{-\lambda t}}{\lambda} + \sum_1^\infty \left(\frac{r}{2r_0}\right)^k \int_0^x e^{-\frac{\lambda+\mu+\xi}{r}(x-y)} \frac{\alpha^k I_{k(\alpha(x-y))}}{(x-y)} \\
&\quad \times \left[\frac{y^k e^{-\frac{\lambda y}{r_0}}}{k!} H(t - \frac{x-y}{r} - \frac{y}{r_0}) q_0(t - \frac{x-y}{r} - \frac{y}{r_0})\right. \\
&\quad \left. + \frac{r_0^k e^{-\lambda(t-\frac{x-y}{r})}(t - \frac{x-y}{r})^k}{k!}\right] dy,
\end{aligned}
$$

$$(4.22)$$

$$
\begin{aligned}
&F_j(t,x) \\
&= \left(\frac{r}{2\mu}\right)^j \sum_0^\infty \left(\frac{r}{2r_0}\right)^k \int_0^x e^{-\frac{\lambda+\mu+\xi}{r}(x-y)} \frac{(j+k)\alpha^{j+k} I_{j+k}(\alpha(x-y))}{k!(x-y)} \left(y^k e^{-\lambda y/r_0}\right. \\
&\quad \times H\left(t - \frac{x-y}{r} - \frac{y}{r_0}\right) q_0\left(t - \frac{x-y}{r} - \frac{y}{r_0}\right) \\
&\quad \left. + r_0^k\, e^{-\lambda(t-\frac{x-y}{r})} \left(t - \frac{x-y}{r}\right)^k\right) dy, \quad j = 1,2,3,\ldots
\end{aligned}
$$

$$(4.23)$$

Particular Case 2: When there is no catastrophe (i.e. $\xi \to 0$) we obtain

$$
\begin{aligned}
F_0(t,x) &= e^{\frac{-\lambda x}{r_0}} q_0(t - \frac{x}{r_0}) + e^{-\lambda t} - e^{\frac{-\lambda x}{r_0}} e^{-\lambda(t - \frac{x}{r_0})} \\
&+ \sum_1^\infty \left(\frac{r}{2r_0}\right)^k \int_0^x e^{-\frac{\lambda+\mu}{r}(x-y)} \frac{\alpha^k k I_{k}(\alpha(x-y))}{(x-y)} \\
&\times \left[\frac{y^k e^{-\frac{\lambda y}{r_0}}}{k!} H(t - \frac{x-y}{r} - \frac{y}{r_0}) q_0(t - \frac{x-y}{r} - \frac{y}{r_0}) \right. \\
&\left. + \frac{r_0^k e^{-\lambda(t - \frac{x-y}{r})}(t - \frac{x-y}{r})^k}{k!} \right] dy
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
F_j(t,x) &= \left(\frac{r}{2\mu}\right)^j \sum_0^\infty \left(\frac{r}{2r_0}\right)^k \int_0^x e^{-\frac{\lambda+\mu}{r}(x-y)} \frac{(j+k)\alpha^{j+k} I_{j+k}(\alpha(x-y))}{k!(x-y)} \\
&\times \left[y^k e^{-\frac{\lambda y}{r_0}} H(t - \frac{x-y}{r} - \frac{y}{r_0}) q_0(t - \frac{x-y}{r} - \frac{y}{r_0}) \right. \\
&\left. + r_0^k e^{-\lambda(t - \frac{x-y}{r})}(t - \frac{x-y}{r})^k \right] dy \quad j = 1, 2, 3, \ldots.
\end{aligned}
\tag{4.25}
$$

Our results coincide with the results in Parthasarathy, Sericola and Vijayashree [11].

Remark. As t tends to infinity, the transient-state system governed by the system of equations tends to steady-state system of differential equations and a stationary solution is thus obtained. The transient distribution of the buffer content is given by

$$
Pr(C(t) > x) = 1 - \sum_0^\infty F_j(t,x).
$$

5. Steady State Solution

In this section, we obtain the steady state for our fluid queueing model. As t tends to infinity, the transient probabilities given by (4.12) and (4.13) tend to stationary solution as follows:

Theorem 5.1.

$$F_0(x)$$
$$= ae^{\frac{-\lambda x}{r_0}} + \frac{\eta\xi}{\lambda(\xi+\eta)}\left[1 + e^{-\lambda x/r_0}\right]$$
$$+ a\sum_1^\infty (\frac{r}{2r_0})^k \int_0^x e^{-\frac{\lambda+\mu+\xi}{r}(x-y)} \frac{\alpha^k k I_{k(\alpha(x-y))}}{(x-y)} \frac{y^k e^{-\lambda y/r_0}}{k!} dy, \qquad (5.1)$$

$$F_j(x) = a(\frac{r}{2\mu})^j \sum_0^\infty (\frac{r}{2r_0})^k \int_0^x e^{-\frac{\lambda+\mu+\xi}{r}(x-y)} \frac{(j+k)\alpha^{j+k}I_{j+k}(\xi(x-y))}{k!(x-y)} y^k e^{-\lambda y/r_0} dy$$
$$j = 1, 2, 3, \ldots. \qquad (5.2)$$

Proof. As t tends to infinity and when we assign $\lim_{t\to\infty} q_0(t) = F_0(0) = a$, the transient probabilities given by (4.12) and (4.13) tend to the above results. $\qquad\square$

Particular Case 3: The above results coincide with the results of Parthasarathy, Vijayashree and Lenin [10] discussed for stationary solution when there is no catastrophe and restoration takes place.

6. Conclusion

In this paper, a fluid model driven by an $M/M/1$ queue with catastrophes with restoration time is discussed. We set up a system of differential equations satisfied by the fluid model. The Laplace transform of the transient-state probability distribution of the fluid queue is obtained through continued fractions. Transient solution for M/M/1 fluid queue is obtained and the corresponding steady state results are obtained explicitly. We also analyze the system when there are no catastrophes and restoration and hence confirmed the respective results obtained by other researchers.

References

[1] I.J.B.F. Adan and J.A.C.Resing, Simple analysis of a fluid queue driven by an M/M/1 queue, *Queueing Systems,* **22** (1996), 171-174.

[2] D. Anick, D. Mitra and M.M. Sondhi, Stochastic theory of a data-handling system with multiple sources, *Bell Syst. Tech. J,* **61** (1982), 1871-1894.

[3] W.G. Coffman, B.M. Igelnik and Y.A. Kogman, Controlled stochastic model of a communication system with multiple sources, *IEEE Trans. Inform. Theory,* **37** (1991), 1379-1387.

[4] A. Elvalid and D. Mitra, Effective bandwidth of general Markovian sources and admission control of high-speed networks, *IEEE/ACM Transactions on Networking,* **1** (1993), 329-343.

[5] D.P. Gaver and J.P. Lehoczky, Channels that cooperatively service a data stream and voice messages, *IEEE Trans. Commun.,* **30** (1982), 1153-1162.

[6] N.K. Jain and Rakesh Kumar, A catastrophic queueing model with correlated input for the cell traffic generated by new broadband services, *Indian J. of Mathematics and Math. Sciences,* **1**, No 2 (2005), 99-108.

[7] N.K. Jain and Rakesh Kumar, Tansient solution of catastrophic-cum-restorative queueing propblem with correlated arrivals and variable service capacity, *Information and Management Sciences,* **18** (2007), 461-465.

[8] S.H. Low and P.P. Varaiya, Burst reducing servers in ATM, *Networks,* **20** (1995), 61-84.

[9] D. Mitra, Stochastic theory of a fluid model of producers and consumers coupled by a buffer, *Adv. in Appl. Prob.,* **20** (1988), 646-676.

[10] P.R. Parthasarathy, K.V. Vijayashree and R.B. Lenin, An M/M/1 driven Fluid Queue - Continued Fraction Approach, *Queueing Systems,* **42** (2002), 189-199.

[11] P.R. Parthasarathy, B. Sericola and K.V. Vijayashree, Exact transient solution of an M/M/1 driven fluid queue, *Int. J. of Comp. and Math.,* **82** (2004), 659-671.

[12] A. Simonian and J. Virtoma, Transient and stationary distributions for fluid queues and input processes with a density, *SIAM J. Appl. Math.,* **51** (1991), 1732-1739.

[13] E.A. Van Doorn and W.R.W. Scheinhardt, Analysis of birth-death fluid queues, *Proc. of Appl. Math. Workshop* (1996), 13-29.

[14] J. Virtamo and I. Norros, Fluid queue driven by an M/M/1 queue, *Queueing Systems,* **16** (1994), 373-386.