

**A STUDY ON MARKOV EVOLUTION ASSOCIATED
WITH A $M/M/1/\infty$ QUEUING SYSTEM**

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Abstract: In this paper, we proceed to analyze a random evolution associated with the $M/M/1/\infty$ model. The study of $M/M/1/\infty$ queuing system has been made very extensively in the queuing literature. The system arises in a huge variety of problems of communication engineering. The object of this paper is to understand the random evolution of cost function associated with $M/M/1/\infty$ model. This is achieved by identifying the unit cost as the fixed velocities of a random motion of a particle in a straight line.

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1. The Transient Solution by Parthasarathy Method

The transient solution for the $M/M/1/\infty$ model has been obtained in literature through several approaches. We give here a simple method provided by Parthasarathy (1987) [11]. Let $p_n(t)$ be probability that there are n customers in the system at time t . Then, by the routine procedure [8], we have

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$$\begin{aligned} p'_n(t) &= \mu p_{n+1}(t) - (\lambda + \mu)p_n(t) + \lambda p_{n-1}(t), \quad n = 1, 2, \dots, \\ p'_0(t) &= \mu p_1(t) - \lambda p_0(t), \end{aligned} \quad (\text{E})$$

where λ and μ have the usual meaning [11]. We assume that there is exactly a customer in the system at time $t = 0$. Then $p_n(0) = \delta_{n,a}$, $n = 0, 1, 2, \dots$, where

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Let

$$q_n(t) = \begin{cases} e^{(\lambda+\mu)t} [\mu p_n(t) - \lambda p_{n-1}(t)], & n = 1, 2, \dots \\ 0, & n = 0, -1, -2, \dots \end{cases} \quad (1)$$

Then, by using the equations (E), we get

$$q'_n(t) = \begin{cases} \mu q_{n+1}(t) + \lambda q_{n-1}(t), & n = 1, 2, \dots \\ 0, & n = 0, -1, -2, \dots \end{cases} \quad (2)$$

subject to the initial condition

$$q_n(0) = \begin{cases} \mu \delta_{n,a} - \lambda \delta_{n-1,a}, & n = 1, 2, \dots \\ 0, & n = 0, -1, -2, \dots \end{cases} \quad (3)$$

Defining

$$H(s, t) = \sum_{n=-\infty}^{\infty} q_n(t) s^n,$$

we get by using the equations (2) and (3),

$$\frac{\partial H(s, t)}{\partial t} = \left(\frac{\mu}{s} + \lambda s \right) H(s, t) - \mu q_1(t), \quad (4)$$

subject to the condition $H(s, 0) = s^a [\mu(1 - \delta_{0,a}) - \lambda s]$. Keeping $q_1(t)$ as unknown, the equation (4) can be readily solved and we obtain

$$H(s, t) = H(s, 0) \exp \left[\left(\frac{\mu}{s} + \lambda s \right) t \right] - \mu \int_0^t \exp \left[\left(\frac{\mu}{s} + \lambda s \right) (t - u) \right] q_1(u) du. \quad (5)$$

Setting $\lambda = \frac{\alpha\beta}{2}$ and $\mu = \frac{\alpha}{2\beta}$, we have $\alpha = 2\sqrt{\lambda\mu}$ and $\beta = \sqrt{\frac{\lambda}{\mu}}$ so that

$$\exp \left[\left(\frac{\mu}{s} + \lambda s \right) t \right] = \exp \left[\frac{1}{2} \left((\beta s) + \frac{1}{\beta s} \right) (\alpha t) \right] = \sum_{n=-\infty}^{\infty} (\beta s)^n I_n(\alpha t) \quad (6)$$

where $I_n(\alpha t)$, $n = 0, \pm 1, \pm 2, \dots$ are the modified Bessel functions of the first kind given by (see e.g. [10])

$$I_n(u) = \sum_{k=0}^{\infty} \frac{u^{n+2k}}{2^{n+2k} k! (n+k)!}, \quad n > -1; \quad I_{-n}(u) = I_n(u).$$

Substituting (6) in (5), we get

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q_n(t) s^n - s^a [\mu(1 - \delta_{0,a}) - \lambda s] \sum_{n=-\infty}^{\infty} (\beta s)^n I_n(\alpha t) \\ - \mu \int_0^t \sum_{n=-\infty}^{\infty} (\beta s)^n I_n(\alpha(t-u)) q_1(u) du. \end{aligned} \quad (7)$$

Equating the coefficient of s^n in (7) for $n = 1, 2, 3, \dots$, we obtain

$$\begin{aligned} q_n(t) = \mu(1 - \delta_{0,a}) \beta^{n-a} I_{n-a}(\alpha t) - \lambda \beta^{n-a-1} I_{n-a-1}(\alpha t) \\ - \mu \beta^n \int_0^t I_n(\alpha(t-u)) q_1(u) du. \end{aligned} \quad (8)$$

In the same manner [12], equating the coefficients of s^n in (7) for $n = -1, -2, \dots$ and using the definition $q_n(t) = 0$, $n = -1, -2, \dots$, we get

$$\begin{aligned} 0 = \mu(1 - \delta_{0,a}) \beta^{n-a} I_{n-a}(\alpha t) - \lambda \beta^{n-a-1} I_{n-a-1}(\alpha t) \\ - \mu \beta^n \int_0^t I_n(\alpha(t-u)) q_1(u) du. \end{aligned} \quad (9)$$

Canceling β^n and replacing n by $-n$ in (9), we get

$$\begin{aligned} 0 = \mu(1 - \delta_{0,a}) \beta^{-a} I_{-n-a}(\alpha t) - \lambda \beta^{-a-1} I_{-n-a-1}(\alpha t) - \mu \int_0^t I_{-n}(\alpha(t-u)) q_1(u) du, \\ n = 1, 2, \dots \end{aligned} \quad (10)$$

Using the fact that $I_n(y) = I_{-n}(y)$, $n = 1, 2, 3, \dots$ in (10), we obtain

$$\begin{aligned} 0 = \mu(1 - \delta_{0,a}) \beta^{-a} I_{n+a}(\alpha t) - \lambda \beta^{-a-1} I_{n+a+1}(\alpha t) - \mu \int_0^t I_n(\alpha(t-u)) q_1(u) du, \\ n = 1, 2, \dots \end{aligned} \quad (11)$$

From (11), we get

$$\mu \int_0^t I_n(\alpha(t-u))q_1(u)du = \mu(1 - \delta_{0,a})\beta^{-a}I_{n+a}(\alpha t) - \lambda\beta^{-a-1}I_{n+a+1}(\alpha t),$$

$$n = 1, 2, \dots \quad (12)$$

Substituting (12) in (8), we get

$$q_n(t) = \mu(1 - \delta_{0,a})\beta^{n-a}I_{n-a}(\alpha t) - \lambda\beta^{n-a-1}I_{n-a-1}(\alpha t) \\ - \beta^n[\mu(1 - \delta_{0,a})\beta^{-a}I_{n+a}(\alpha t) - \lambda\beta^{-a-1}I_{n+a+1}(\alpha t)], \quad n = 1, 2, 3, \dots \quad (13)$$

Simplifying (13), we get

$$q_n(t) = \mu(1 - \delta_{0,a})\beta^{n-a}[I_{n-a}(\alpha t) - I_{n+a}(\alpha t)] \\ + \lambda\beta^{n-a-1}[I_{n+a+1}(\alpha t) - I_{n-a-1}(\alpha t)], \quad n = 1, 2, 3, \dots \quad (14)$$

Now from (E) we have

$$p'_0(t) = \exp[-(\lambda + \mu)t]q_1(t)$$

and hence by integration we obtain

$$p_0(t) = \int_0^t \exp[-(\lambda + \mu)u]q_1(u)du + \delta_{0,a}. \quad (15)$$

Next, from (1), we have for $n = 1, 2, 3, \dots$,

$$p_n(t) = \frac{\exp[-(\lambda + \mu)t]}{\mu}q_n(t) + \left(\frac{\lambda}{\mu}\right)p_{n-1}(t)$$

and hence by iteration, we have

$$p_n(t) = \frac{\exp[-(\lambda + \mu)t]}{\mu} \sum_{k=1}^n \left(\frac{\lambda}{\mu}\right)^{n-k} q_k(t) + \left(\frac{\lambda}{\mu}\right)^n p_0(t),$$

$$n = 1, 2, 3, \dots \quad (16)$$

2. The Probability Density of a Busy Cycle

A busy period is defined as the time interval between the time of arrival of a customer to an idle server and the time of the server next becoming idle. An idle period is the time interval between the time of server becoming idle and the time of server next becoming busy. The busy period and the idle period are independent. A busy cycle is the sum of the busy period and the adjacent idle period. Hence the probability density function of the busy cycle is the convolution of the probability density of the idle period with that of the busy period [14]. Since the time interval between any two successive arrivals in the $M/M/1/\infty$ model is assumed to follow exponential distribution with mean $\frac{1}{\lambda}$, the probability density function $\phi(t)$ of the idle period is $\lambda \exp[-\lambda t]$. So the probability density of the busy period is sufficient to find the probability density of the busy cycle and it is found as follows.

For the model $M/M/1/\infty$, we impose that there is an absorbing barrier at zero system size and assume that the initial system size is 1. Let $p_n(t)$ be the probability that the system size is n at time t . Then we have

$$p'_0(t) = \mu p_1(t), \quad (17)$$

$$p'_1(t) = -(\lambda + \mu)p_1(t) + \mu p_2(t), \quad (18)$$

$$p'_n(t) = -(\lambda + \mu)p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t), \quad n \geq 2. \quad (19)$$

The equations (17) to (19) are subject to the condition $p_n(0) = \delta_{1,n}$, $n = 0, 1, 2, \dots$. it is clear that $p'_0(t)$ is the probability density function of the busy period. To find it, we proceed as follows. Define

$$r_n(t) = \begin{cases} e^{(\lambda+\mu)t} [\mu p_n(t) - \lambda p_{n-1}(t)], & n \geq 2, \\ e^{(\lambda+\mu)t} \mu p_1(t), & n = 1, \\ 0, & n \leq 0. \end{cases} \quad (20)$$

Then using (17) to (20), we get

$$r'_n(t) = \begin{cases} \mu r_{n+1}(t) + \lambda r_{n-1}(t), & n \geq 2, \\ \mu r_2(t) + \lambda r_1(t), & n = 1, \\ 0, & n \leq 0. \end{cases} \quad (21)$$

The equations (21) are subjected to the initial conditions

$$r_n(0) = \begin{cases} \mu, & n = 1, \\ -\lambda, & n = 2, \\ 0, & \text{otherwise.} \end{cases} \quad (22)$$

We proceed to solve the equation (21) subject to (22). Defining

$$G(s, t) = \sum_{n=-\infty}^{\infty} r_n(t) s^n$$

and using (21), we get

$$\frac{\partial G(s, t)}{\partial t} = \left(\frac{\mu}{s} + \lambda s \right) G(s, t) + (\lambda s - \mu) r_1(t). \quad (23)$$

Solving the equation (22), we get

$$\begin{aligned} G(s, t) = G(s, 0) \exp \left\{ \left(\frac{\mu}{s} + \lambda s \right) t \right\} \\ + (\lambda s - \mu) \int_0^t \exp \left\{ \left(\frac{\mu}{s} + \lambda s \right) (t - u) \right\} r_1(u) du. \end{aligned} \quad (24)$$

But $G(s, 0) = \mu s - \lambda s^2$. Thus, we have

$$\begin{aligned} G(s, t) = (\mu s - \lambda s^2) \exp \left\{ \left(\frac{\mu}{s} + \lambda s \right) t \right\} \\ + (\lambda s - \mu) \int_0^t \exp \left\{ \left(\frac{\mu}{s} + \lambda s \right) (t - u) \right\} r_1(u) du. \end{aligned} \quad (25)$$

Substituting (6) in (25), we get

$$\begin{aligned} \sum_{n=-\infty}^{\infty} r_n(t) s^n = (\mu s - \lambda s^2) \sum_{n=-\infty}^{\infty} (\beta s)^n I_n(\alpha t) \\ + (\lambda s - \mu) \int_0^t \sum_{n=-\infty}^{\infty} (\beta s)^n I_n(\alpha(t - u)) r_1(u) du. \end{aligned} \quad (26)$$

Equating the coefficients of s^0 on both sides of (26), we have

$$\begin{aligned} 0 = \frac{\mu}{\beta} I_{-1}(\alpha t) - \frac{\lambda}{\beta^2} I_{-2}(\alpha t) \\ + \frac{\lambda}{\beta} \int_0^t I_{-1}(\alpha(t - u)) r_1(u) du - \mu \int_0^t I_0(\alpha(t - u)) r_1(u) du. \end{aligned} \quad (27)$$

Using the property $I_n(y) = I_{-n}(y)$ in (27), we get

$$0 = \frac{\mu}{\beta} I_1(\alpha t) - \frac{\lambda}{\beta^2} I_2(\alpha t) + \frac{\lambda}{\beta} \int_0^t I_1(\alpha(t-u)) r_1(u) du - \mu \int_0^t I_0(\alpha(t-u)) r_1(u) du. \quad (28)$$

Solving the equation (28), we get

$$r_1(t) = \frac{I_1(\alpha t)}{\beta t}. \quad (29)$$

Substituting $r_1(t) = e^{(\lambda+\mu)t} \mu p_1(t)$ in (29), we get the probability density function $\psi(t)$ of the busy period given by

$$\psi(t) = p'_0(t) = \mu p_1(t) = \frac{e^{-(\lambda+\mu)t} I_t(\alpha t)}{\beta t}. \quad (30)$$

3. An Associated Cost Function

It is enough to describe the cost function for a busy cycle. Let T be the length of a busy cycle. Let T_1 and T_2 be the corresponding busy and idle periods. Let C_1 be profit per unit time of customer service and C_2 be the loss per unit time of the idle period. Assume that $C_1, C_2 > 0$. Then the net profit $C(T)$ over a busy cycle is given by

$$C(T) = C_1 T_1 - C_2 T_2. \quad (31)$$

It is easy to find the mean of $C(T)$ as follows. Taking expectation on both sides of (31), we have

$$E[C(T)] = C_1 \int_0^\infty u \psi(u) du - C_2 \int_0^\infty y e^{-\lambda y} \lambda dy. \quad (32)$$

4. A Random Motion Analysis

We can consider the cost C_1 as the positive velocity and C_2 as the negative velocity of a particle moving along the real line during busy period and idle period respectively. Then the distance traveled in the busy cycle starting at the origin is given by (31). This can be used to analyze the distance traveled in any time t . Let $\zeta(t)$ be number of times the server entering in to idle state during the interval $(0, t]$. Then, for the case in which $\zeta(t) = 0$, we have

$$C(t) = C_1 t \quad (33)$$

and

$$P_r[\zeta(t) = 0] = 1 - \int_0^t \psi(u) du. \quad (34)$$

For the case in which $\zeta(t) = 1$, we have

$$E[C(t)] = \int_0^t [uC_1 - (t-u)C_2] \psi(u) e^{-\lambda(t-u)} du \quad (35)$$

and

$$P_r[\eta(t) = 1] = \int_0^t \psi(u) e^{-\lambda(t-u)} du. \quad (36)$$

Proceeding in the same manner, we have for the case $\zeta(t) = 2$,

$$E[C(t)] = \int_0^t \int_u^v \int_v^t [(w-v+u)C_1 - (t-w+v-u)C_2] \psi(u) \lambda e^{-\lambda(v-u)} \psi(w-v) e^{-\lambda(t-w)} du dv dw \quad (37)$$

and

$$P_r[\eta(t) = 2] = \int_0^t \int_u^v \int_v^t \psi(u) e^{-\lambda(u-v)} \lambda \psi(w-v) e^{-\lambda(t-w)} du dv dw. \quad (38)$$

5. Conclusion

The random evolution of the cost function associated with $M/M/1/\infty$ queuing system is unique. It is achieved by identifying the unit cost as the fixed velocities of a random motion of a particle in the straight line.

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