International Journal of Applied Mathematics

Volume 26 No. 1 2013, 83-91
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v26i1.8

ON WEAK DOMINATION IN A ZERO DIVISOR GRAPH

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Abstract: Let R be a commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of R. The zero-divisor graph of a ring is the graph(simple) whose vertex set is the set of non-zero zero-divisors, and an edge is drawn between two distinct vertices if their product is zero. For a graph $\Gamma(Z_n)$, a set $S \subseteq V(\Gamma(Z_n))$ is a weak dominating set if every vertex v in $V(\Gamma(Z_n)) - S$ has a neighbour u in S such that the degree of u is not greater than the degree of v. The minimum cardinality of a weak dominating set of $\Gamma(Z_n)$ is the weak domination number, $\gamma_w(\Gamma(Z_n))$.

In this paper, we present some bounds on $\Gamma(Z_n)$ and give exact values for $\gamma_w(\Gamma(Z_{2p}))$, $\gamma_w(\Gamma(Z_{p^2}))$, $\gamma_w(\Gamma(Z_{pq}))$, $\gamma_w(\Gamma(Z_{2^n p}))$, $\gamma_w(\Gamma(Z_{3^n p}))$ and $\gamma_w(\Gamma(Z_{pqr}))$.

AMS Subject Classification: 05C25, 05C69
Key Words: commutative ring, zero divisor graph, weak domination

1. Introduction

Let R be a commutative ring and let $Z(R)$ be its set of zero-divisors. The zero-divisor graph of a ring is the graph(simple) whose vertex set is the set of non-zero zero-divisors, and an edge is drawn between two distinct vertices if their product is zero. Throughout this paper, we consider the commutative ring by R and zero divisor graph $\Gamma(R)$ by $\Gamma(Z_n)$. The idea of a zero-divisor

Received: January 18, 2013

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graph of a commutative ring was introduced by I. Beck in [2], where he was mainly interested in colorings. The zero divisor graph is very useful to find the algebraic structures and properties of rings.

In [9], Sampathkumar and Pushpa Latha have introduced the concept of weak domination in graphs. The weak domination number of $\Gamma(Z_n)$, denoted by $\gamma_w(\Gamma(Z_n))$ is defined by, $\Gamma(Z_n) = \{|S|$, where $S \subseteq V(\Gamma(Z_n))$, $S \neq \phi, N(S) \neq V(\Gamma(Z_n))\}$ which satisfies the following conditions; (i) $N(S) \cup S = V(\Gamma(Z_n))$ (ii) $N(S) \cap S = \phi$ (iii) $d(u) \leq d(v)$ for $u \in S$ and $v \in N(S)$ (iv) no two vertices in S are adjacent. The weak domination number is the minimum cardinality of a weak dominating set of $\Gamma(Z_n)$. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V/uv \in E\}$. In this paper, we give upper bounds of weak domination number of zero divisor graphs. For notation and graph theory terminology, we in general follow [3], [4], [5] and the structure of the zero divisor graph is [1], [6], [7], [8].

2. Weak Domination Number of $\Gamma(Z_n)$

Theorem 1. *A graph $\Gamma(Z_n)$ has a domination set iff $\Gamma(Z_n)$ is connected and n is a composite number.*

Theorem 2. *For any graph $\Gamma(Z_{2p})$, where p is any prime number, then $\gamma_w(\Gamma(Z_{2p})) = p - 1$.*

Proof. The vertex set of $\Gamma(Z_{2p})$ is $\{2, 4, 6, ..., 2(p-1), p\}$. Let $u = 4$ and $v = p$ then $2p$ must divides uv. That is $2p$ divides $4p$. Clearly, u and v are adjacent vertices. Similarly, any vertex u in $V(\Gamma(Z_{2p}))$ and $v = p$ then $2p$ must divides uv. It seems that p is adjacent to all the vertices in $V(\Gamma(Z_{2p}))$. Let $u = 4 \neq p$ and $w = 6 \neq p$ in $V(\Gamma(Z_{2p}))$ such that $uw \neq 0$. It means that $2p$ does not divide $uv = 24$. Clearly, no two vertices in $\Gamma(Z_{2p})$ are adjacent, except p. Using theorem, (3.2) in [7], $\Gamma(Z_{2p})$ is a star graph.

Let S be a weak dominating set of $\Gamma(Z_{2p})$. Since, $|V(\Gamma(Z_{2p}))| = p$. Let us assume that $\gamma_w(\Gamma(Z_{2p})) = 2$ with $p \geq 3$. It is enough to show that $\Gamma(Z_{2p})$=$K_{1,p-1}$. If $p = 3$, then $\Gamma(Z_{2p})$=$K_{1,2}$. Assume that $p \geq 5$ and $\Gamma(Z_{2p}) \neq K_{1,p-1}$. This means that there exists a path of length at least three in $\Gamma(Z_{2p})$. Let $uvwz$ be a path in $\Gamma(Z_{2p})$, with u and z are pendant vertices. Since, $\Gamma(Z_{2p})$ is a connected graph with $p \geq 3$ vertices and let P be the set of pendant vertices of $\Gamma(Z_{2p})$ then P⊆S. Clearly any vertex $u \in S$ and $v \in V(\Gamma(Z_{2p})) - S$ then $deg(u) \leq deg(v)$. If u is pendant vertex then, $1 = deg(u) \leq deg(v)$, which implies, $u, z \in S$. Thus, it follows that v is weekly dominated by u and w is weekly dominated by z. Consequently, $\{u, v, z\} \subseteq S$ or $\{u, w, z\} \subseteq S$ or $\{u, v, w, z\} \subseteq S$ or $\{u, z\} \subseteq S$. This, contradicts our assumption $\gamma_w(\Gamma(Z_{2p}))$=2. Hence, $\Gamma(Z_{2p})$=$K_{1,p-1}$ and $\gamma_w(\Gamma(Z_{2p})) = p - 1$. □

Theorem 3. *If p is any prime > 2 then, $\Gamma(Z_{2p}) = K_{1,p-1}$ if and only if* $\gamma_w(\Gamma(Z_{2p})) = \Delta(\Gamma(Z_{2p}))$.

Proof. The theorem is true for order 2 or 3. So, we shall suppose that $\Gamma(Z_{2p}$ has at least 4 vertices. Let us assume that $\Gamma(Z_{2p}) = K_{1,p-1}$. Let u be a support vertex of $\Gamma(Z_{2p})$ that is adjacent to two (or more) end vertices that does not belong to a weak dominating set. Let E_u denote the set of edges incident with u and S be a minimum weak dominating set for $\Gamma(Z_{2p}) - E_u$. Then u is in S and S/u is a weak dominating set for $\Gamma(Z_{2p})$. Clearly, $\gamma_w(\Gamma(Z_{2p} - E_u)) \geq \gamma_w(\Gamma(Z_{2p})$. Hence, $\gamma_w(\Gamma(Z_{2p})) = |E_u| = d(u) = \Delta(\Gamma(Z_{2p}))$.

Conversely, let $\gamma_w(\Gamma(Z_{2p})) = \Delta(\Gamma(Z_{2p}))$. Using Theorem 2.2, $\Gamma(Z_{2p})$ is a star graph, namely $K_{1,p-1}$. □

Theorem 4. *If p is any prime number, then* $\gamma_w(\Gamma(Z_{p^2})) = 1$.

Proof. If p is any prime, then $V(\Gamma(Z_{p^2})) = \{p, 2p, 3p, 4p, ..., (p-1)p\}$. Clearly p is adjacent to all the vertices in $\Gamma(Z_{p^2})$. Also note that, any two vertices in $\Gamma(Z_{p^2})$ is adjacent and hence $\Gamma(Z_{p^2})$ is a complete graph, namely K_{p-1}.

Let W be a subgraph of $\Gamma(Z_{p^2})$ defined by $\Gamma(Z_{p^2}) - e = W$. Clearly, W is obtained by removing an edge from $\Gamma(Z_{p^2})$. Since, the degree of all the vertices in $\Gamma(Z_{p^2})$ is $p - 2$. That is, $d(u) = d(v) = p - 2$, where $u, v \in V(\Gamma(Z_{p^2}))$. Let e be an edge from a vertex u to a vertex v, then $d(u) = d(v) = p - 3$, where $u, v \in W = V(\Gamma(Z_{p^2}) - e)$. It seems that W contains two vertices of degree $p-3$. Let, S be a weak domination set of W, then $S = u, v$ and $d(w) > d(u) = d(v)$, where any vertex $w \in V(\Gamma(Z_{p^2}))$. Thus, $\gamma_w(W) = |S| = 2 > 1 = \gamma_w(W + e) = \gamma_w(\Gamma(Z_{p^2}))$. □

Theorem 5. *If p and q are distinct prime numbers with $p < q$, then* $\gamma_w(\Gamma(Z_{pq}) = q - 1$.

Proof. The proof is by the method of induction on p and q. The vertex set of $\Gamma(Z_{pq})$ is $\{p, 2p, 3p, ..., p(q-1), q, 2q, 3q, ..., (p-1)q\}$.

Case (i) Let $p = 2$, q is any prime > 2.
Using theorem (3.1), $\gamma_w(\Gamma(Z_{2q}) = q - 1$.

Case (ii) Let $p = 3$, q is any prime > 3.
The vertex set of $\Gamma(Z_{3q})$ is $\{3, 6, 9, ..., 3(q-1), q, 2q\}$. Let u and v are two vertices in $\Gamma(Z_{3q})$ with maximum degree. Let $u = q$ and $v = 2q$, then there exist any other vertex $w \neq q \neq 2q$ in $\Gamma(Z_{3q})$ such that w is adjacent to both u and v. That is, $uw = vw = 0$. But $uv = 2q^2$ which does not divide by $3q$. Therefore u and v are non-adjacent vertices. Then the vertex set V can be partition into two parts V_1 and V_2 such that $V_1 = \{u, v\} = \{q, 2q\}$ and $V_2 = \{3, 6, 9, ..., 3(q-1)\}$. Clearly $|V_1| = 2$ and $|V_2| = q - 1$, then $|V| = |V_1| + |V_2| = 2 + q - 1 = q + 1$. Note that the vertices in the second partite set have the smallest degree. Since, $2 < q-1$, then to weakly dominate these vertices, we need include all of them in any weakly dominating set. Clearly, every vertex in V_2 which weakly dominate all the vertices in V_1. Hence, $\gamma_w(\Gamma(Z_{3q})) = |V_2| = q - 1$.

Case (iii) Let $p = 5$, q is any prime > 5.
The vertex set of $\Gamma(Z_{5q})$ is $\{5, 10, ..., 5(q-1), q, 2q, 3q, 4q\}$. Clearly, number of vertices in $\Gamma(Z_{5q}) = q + 3$. Let u and v be any two vertices in $\Gamma(Z_{5q})$ with maximum and minimum degree, respectively. Let $u = q$ and $v = 10$, then $5q$ must divide uv which implies that u and v are adjacent. Let $u = q$ and $w = 2q$ then $5q$ does not divide $uw = 2q^2$, which implies that u and w are non-adjacent vertices. Then the vertex set V can be partitioned into two parts V_1 and V_2, where $V_1 = \{q, 2q, 3q, 4q\}$ and $V_2 = \{5, 10, ..., 5(q-1)\}$. Clearly any two vertices in V_1 are non-adjacent as same as V_2. Finally we note that, every vertex in V_1 is adjacent to all the vertices in V_2. Moreover $V(\Gamma(Z_{5q})) = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$. It seems that the vertices in the second partite set have the smallest degree. Because, $|V_2| > |V_1|$ implies that every vertex in V_2 is dominating all the vertices in V_1 and hence $\gamma_w(\Gamma(Z_{5q})) = |V_2| = q - 1$.

Case (iv) Let $p < q$.
The vertex set of $\Gamma(Z_{pq})$ is $\{p, 2p, 3p, ..., p(q-1), q, 2q, 3q, ..., (p-1)q\}$. Let $v = p$ and $w = p(q-1)$ in $\Gamma(Z_{pq})$ the pq does not divides $uw = p^2(q-1)$. Clearly v and w are non-adjacent vertices. Let $u = q$ and $v = p$ then pq must divides uv, which implies that u and v are adjacent vertices. So the vertex set V can be partition into two parts V_1 and V_2 which implies that the vertex p, multiples of p are in V_1 and q, multiples of q are in V_2. Clearly every vertices in V_1 are non-adjacents same as V_2. Then, $|V| = |V_1| + |V_2| = p - 1 + q - 1 = p + q - 2$. Since $p < q$, then $|V_1| < |V_2|$. Clearly, $d(u) < d(v)$ where $u \in V_1$ and $v \in V_2$. Let $v \in V_1$ then by removing all edges incident with v, we obtain a graph H containing two components K_1 and $K_{p-2,q-1}$. Hence,

$\gamma_w(H) = \gamma_w(K_1) + \gamma_w(K_{p-2,q-1}) = 1 + q - 1 > q - 1 = \gamma_w(\Gamma(Z_{pq})).$ □

Theorem 6. *For any graph $\Gamma(Z_{2^n})$, where $n > 2$ is a positive integer, then*

a) If n is even, $\gamma_w(\Gamma(Z_{2^n})) = 2^{\frac{n}{2}} \sum_{i=0}^{\frac{n-4}{2}} 2^i + 1.$

b) Otherwise, $\gamma_w(\Gamma(Z_{2^n})) = 2^{\frac{n-1}{2}} \sum_{i=0}^{\frac{n-3}{2}} 2^i.$

Proof. The vertex set of $\Gamma(Z_{2^n})$ is $\{2, 4, .., 2(2^{n-1} - 1)\}$ and $|V(\Gamma(Z_{2^n}))| = 2^{n-1} - 1$. The proof is by the method of induction on n.

Case (a) n is even.

Subcase (i) Let $n = 4$.

The vertex set of $\Gamma(Z_{2^4})$ is $\{2, 4, 6, 8, 10, 12, 14\}$. Let S be a vertex subset of V such that $d(u) \leq d(v)$, where $u \in S$ and $v \in N(S)$. Let P be a set of all pendant vertices in $\Gamma(Z_{2^4})$. Clearly, $P = \{2, 6, 10, 14\}$ with $d(u) = 1$, for all $u \in P$. It seems that $P \subseteq S$. Let $v = 2^{n-1} = 2^{4-1} = 8$ and w be any other vertex in $\Gamma(Z_{2^4})$. Suppose $w = 2^4 - 2$, then $vw = 8 \times (2^4 - 2) = 112$. Clearly, 2^4 must divides 112. Thus, the vertex v is adjacent to all vertices in $\Gamma(Z_{2^4})$ which implies $v = 8 \in N(S)$. Let $x = 4$ and $y = 12$ be the remaining vertices in V such that $xv = yv = 0$. That is, x, y and v are adjacent vertices. Clearly, either $x = 4 \in S$ or $y = 12 \in S$. Suppose, $x, y \in S$, we get a contradiction for our definition that no two vertices in S are adjacent. Finally we conclude that $S = \{2, 4, 6, 10, 14\}$ or $S = \{2, 6, 10, 12, 14\}$ and $N(S) = \{8, 12\}$ or $N(S) = \{4, 8\}$, respectivily. Since, degree of any vertex in S is less than or equal to degree of any vertex in $N(S)$ and no two vertices in S are adjacent which implies that $\gamma_w(\Gamma(Z_{2^{n=4}})) = |S| = 5 = 2^2 + 2^0 = 2^{4/2}(2^0) + 2^0 = 2^{n/2} \sum_{i=0}^{\frac{n-4}{2}} 2^i + 1.$

Subcase (ii) Let $n = 6$.

The vertex set of $\Gamma(Z_{2^6})$ is $\{2, 4, 6, ..., 62\}$. Let S be a vertex subset of V such that $d(u) \leq d(v)$, where $u \in S$ and $v \in N(S)$. Let P be a set of all pendant vertices in $\Gamma(Z_{2^6})$. Clearly, $P = \{2, 6, ..., (2^6 - 2)\}$ with $d(u) = 1$, for all $u \in P$. It seems that $P \subseteq S$. Using subcase (i), let $v = 2^{n-1} = 2^{6-1} = 32$ and $w = 2^6 - 2$ be any other vertex in $\Gamma(Z_{2^6})$ such that 2^6 must divides $vw = 32 \times (2^6 - 2) = 1984$. Thus, the vertex v is adjacent to all vertices in $\Gamma(Z_{2^6})$ which implies $v = 32 \in N(S)$. Similarly, 2^4 and 3×2^4 are adjacent to all the vertices in $\Gamma(Z_{2^6})$ except P, then $16, 48 \in N(S)$.

Let U be a vertex subset of V with $U = \{4, 12, 20, ..., (2^6 - 4)\}$. Clearly, no two vertices in U is adjacent and every vertex in U is adjacent to $\{16, 32, 48\}$. It seems that $d(U) < d(N(S))$ which implies that $U \subseteq S$.

Let $W = V - (P \cup U \cup N(S)) = \{8, 24, 40, 56\}$ be a vertex subset of V. Finally, we obtain that the vertices in W make a complete graph, namely K_4 and all the vertices in W are adjacents to $N(S)$. Using theorem (2.4), any one of the vertex in W is in S. Otherwise, if any two vertices in W belongs to S, then we get a contradiction that no two vertices are adjacent in S.

Hence, $\gamma_w(\Gamma(Z_{2^6})) = |S| = |P| + |U| +$any one vertex in W.

$$= 16 + 8 + 1 = 25 = 2^4 + 2^3 + 2^0 = 2^3(2^1 + 2^0) + 1$$

$$= 2^{6/2} \sum_{i=0}^{1} 2^i + 1 = 2^{n/2} \sum_{i=0}^{\frac{n-4}{2}} 2^i + 1, \text{ where } n = 6.$$

Subcase(iii): Let $n > 6$ is even.

The vertex set of $\Gamma(Z_{2^n})$ is $\{2, 4, ..., 2^{n-1}, 2(2^{n-1} - 1)\}$ and $|V(\Gamma(Z_{2^n}))| = 2^{n-1} - 1$. Since P is a pendant vertex set with $|P| = 2^{n-2}$. Using above cases,

$$\gamma_w(\Gamma(Z_{2^n})) = |S| = 2^{n-2} + ... + 2^{\frac{n}{2}} + 2^0$$

$$= 2^{\frac{n}{2}}(2^0 + ... + 2^{\frac{n}{2}-1}) + 2^0$$

$$= 2^{\frac{n}{2}} \sum_{i=0}^{\frac{n-4}{2}} 2^i + 1, \text{ where } n \text{ is even.}$$

Case (b) n is odd.

Subcase (i) Let $n = 3$.

The vertex set of $\Gamma(Z_{2^3})$ is $\{2, 4, 6\}$. Let S be a vertex subset of V such that $d(u) \leq d(v)$, where $u \in S$ and $v \in N(S)$. Let P be a set of all pendant vertices in $\Gamma(Z_{2^3})$. Clearly, $P = \{2, 6\}$ with $d(u) = 1$, for all $u \in P$. It seems that $P \subseteq S$. Let $v = 6$ and w be any other vertex in $\Gamma(Z_{2^3})$. Suppose $w = 2$, then $vw = 8$. Clearly, 2^3 must divides 8. Thus, the vertex v is adjacent to all vertices in $\Gamma(Z_{2^3})$ which implies $v = 4 \in N(S)$. Let $x = 2$ and $y = 6$ be the remaining vertices in V such that $xv = yv = 0$ and $xy \neq 0$. Finally we conclude that $S = \{2, 6\}$ and $N(S) = \{4\}$. Hence, $\gamma_w(\Gamma(Z_{2^{n=3}})) = |S| = 2 = 2^{(3-1)/2}(2^0) = 2^{(n-1)/2} \sum_{i=0}^{\frac{n-3}{2}} 2^i$, where $n = 3$.

Subcase (ii) Let $n = 5$.

The vertex set of $\Gamma(Z_{2^5})$ is $\{2, 4, ..., 30\}$. Let P be a set of all pendant vertices in $\Gamma(Z_{2^5})$. Clearly, $P = \{2, 6, ..., 30\}$ with $d(u) = 1$, for all $u \in P$. It seems that $P \subseteq S$. Let $v = 16$ and w be any other vertex in $\Gamma(Z_{2^5})$. Suppose $w = 2$, then $vw = 32$. Clearly, 2^5 must divides 32. Thus, the vertex v is adjacent to all vertices in $\Gamma(Z_{2^5})$ which implies $v = 16 \in N(S)$. Let U be a vertex subset of V with $U = \{4, 8, 12, 20, 24, 28\}$. Let $x = 8$ and $y = 24$ be two vertices in U is adjacent to remaining vertices in U. Clearly, $d(4) = d(12) = d(20) = d(28) < d(8) = d(24)$ implies that the vertices $8, 24 \in N(S)$ and the remaining vertices $4, 12, 20, 28 \in S$. Therefore the set $S = \{2, 4, 6, 10, 12, 14, 18, 20, 22, 26, 28, 30\}$. Hence, $\gamma_w(\Gamma(Z_{2^{n=5}})) = |S| = 12 = 2^3 + 2^2 = 2^2(2^1 + 2^0) = 2^{(5-1)/2}(2^1 + 2^0) = 2^{(n-1)/2} \sum_{i=0}^{\frac{n-3}{2}} 2^i$, where $n = 5$.

Subcase (iii) Let $n > 5$ is any odd number.

Using above two subcases, $\gamma_w(\Gamma(Z_{2^n})) = 2^{(n-1)/2} \sum_{i=0}^{\frac{n-3}{2}} 2^i$, where n is odd number. □

Theorem 7. *If $p > 4$ is any prime, then $\gamma_w(\Gamma(Z_{4p})) = 2(p-1)$.*

Proof. The vertex set of $\Gamma(Z_{4p})$ is

$$\{2, 4, 6, 8, ..., 2(2p-1), p, 2p, 3p\}$$

with $|V(\Gamma(Z_{4p}))| = 2p + 1$. Let $v = 2p$ be a vertex and let w be any vertex such that $4p$ divides vw. Clearly, v is adjacent to all the vertices in $V(\Gamma(Z_{4p}))$. Let $P, S, N(S)$ be the pendant set, minimum degree set, neighbourhood os S, respectively. Since v has maximum degree then $v \in N(S)$.

Case (i) Let $p = 5$.

The vertex set of $\Gamma(Z_{20})$ is $\{2, 4, .., 2(10-1), 5, 10, 15\}$ with $|V(\Gamma(Z_{20}))|=$ 11. Let $v = 2p = 10$ be a vertex and let w be any vertex such that 20 divides vw. Clearly, $v = 10$ is adjacent to all the vertices in $V(\Gamma(Z_{20}))$ then $10 \in N(S)$. Let $x = 2$ and $y = 14$ then 28 is not divisible by 20 which implies x and y are non adjacent vertices. But $xv = yv = 0$. Then, the pendant set $P = \{2, 6, 14, 18\}$ with degree of any vertex in P is 1 and $P \subseteq S$.

Let $U = \{4, 8, 12, 16\}$ be a vertex subset of $V(\Gamma(Z_{20}))$. Clearly no two vertices in U is adjacent. That is 20 does not divide $32(= 4 \times 8)$. It means that no edge conneted between the vertices 4 and 8. But, the vertices in U are adjacent to the vertices $5, 10$, and 15 with $d(4) = d(8) = d(12) = d(16) < d(5) = d(15)$. Clearly, $U \subseteq S$ and the vertices $5, 15 \in N(S)$.

Hence, $\gamma_w(\Gamma(Z_{20})) = |S| = |P| + |U| = 4 + 4 = 8 = 2 \times 5 - 2 = 2(p-1)$, where $p = 5$.

Case (ii) Let $p = 7$.

The vertex set of $\Gamma(Z_{28})$ is $\{2, 4, .., 2(14-1), 7, 14, 21\}$ with $|V(\Gamma(Z_{28}))|=$ $2p + 1 = 15$. Let $v = 2p = 14$ be a vertex and let w be any vertex such that 28 divides vw. Clearly, $v = 14$ is adjacent to all the vertices in $V(\Gamma(Z_{28}))$ then $14 \in N(S)$. Let $x = 6$ and $y = 18$ then 108 is not divisible by 28 which implies x and y are non adjacent vertices. But $xv = yv = 0$. Then, the pendant set $P = \{2, 6, 10, 18, 22, 26\}$ with degree of any vertex in P is 1 and $P \subseteq S$.

Let $U = \{4, 8, 12, 16, 20, 24\}$ be a vertex subset of $V(\Gamma(Z_{20}))$. Clearly no two vertices in U is adjacent. That is 28 does not divide $96(= 8 \times 12)$. But, the vertices in U are adjacent to the vertices $7, 14$, and 21 with $d(4) = d(8) =$ $.... = d(24) < d(7) = d(21)$. Clearly, $U \subseteq S$ and the vertices $7, 21 \in N(S)$.

Hence, $\gamma_w(\Gamma(Z_{20})) = |S| = |P| + |U| = 6 + 6 = 12 = 2 \times 7 - 2 = 2(p-1)$, where $p = 7$.

Case (iii) Let $p > 7$.

The vertex set of $\Gamma(Z_{4p})$ is $\{2, 4, ..., 2(2p-1), p, 2p, 3p\}$ with $|V(\Gamma(Z_{4p}))| = 2p + 1$. Let $v = 2p$ be a vertex and let w be any other vertex such that $4p$ divides vw. Clearly, v is adjacent to all the vertices in $V(\Gamma(Z_{4p}))$ and $v = 2p \in N(S)$. Let P be the pendant vertex set and using above cases, $P = \{2, 6, ..., 2(p-2), 2(p+2), ..., 2(2p-1)\}$. Similarly, Let $U = \{4, ..., 4(p-1)\}$. Since, no two vertices in U is adjacent. That is, $4p$ does not divide $32(p-1)(= 8 \times 4(p-1))$. But, the vertices in U are adjacent to the vertices $p, 2p$, and $3p$ with $d(4) = d(8) = ... = d(4(p-1)) < d(p) = d(3p)$. Clearly, $U \subseteq S$ and the vertices $p, 3p \in N(S)$ which implies that the vertex set of $N(S)$ is $\{p, 2p, 3p\}$.

Hence, $\gamma_w(\Gamma(Z_{4p})) = |S| = |P| + |U| = |V(\Gamma(Z_{4p}))| - |N(S)| = 2p + 1 - 3 = 2p - 2 = 2 \times 7 - 2 = 2(p-1)$, where p is any prime. □

Theorem 8. *In $\Gamma(Z_{8p})$ where $p > 8$ is any prime, then $\gamma_w(\Gamma(Z_{8p})) = 4(p-1)$.*

Since, the vertex set of $\Gamma(Z_{8p})$ is $\{2, 4, ..., 2(4p-1), p, 2p, 3p, 4p, 5p, 6p, 7p\}$ with $|V(\Gamma(Z_{8p}))| = 4p + 3$. Using Theorem 2.7, $N(S) = \{p, 2p, 3p, ..., 7p\}$ and $|N(S)| = 7$. Hence, $\gamma_w(\Gamma(Z_{8p})) = |V(\Gamma(Z_{8p}))| - |N(S)| = 4p + 3 - 7 = 4p - 4 = 4(p-1)$.

Theorem 9. *In $\Gamma(Z_{2^n p})$ where $p > 2^n$ is any prime and n is any positive integer, then $\gamma_w(\Gamma(Z_{2^n p})) = 2^{n-1}(p-1)$.*

Proof. The vertex set of $\Gamma(Z_{2^n p})$ is $\left\{2, .., 2(2^{n-1}p - 1), p, 2p, ..., (2^n - 1)p\right\}$ with $|V(\Gamma(Z_{2^n p}))| = 2^{n-1}p + 2^{n-1} - 1$. Using Theorems 2.7 and 2.8, $N(S) = \{p, 2p, ..., (2^n - 1)p\}$.

Hence, $\gamma_w(\Gamma(Z_{2^n p})) = |V(\Gamma(Z_{2^n p}))| - |N(S)| = 2^{n-1}p + 2^{n-1} - 1 - (2^n - 1) = 2^{n-1}(p-1)$. □

Theorem 10. *For any prime $p > 3$, $\gamma_w(\Gamma(Z_{3^n})) = 3^{n-1} - 8$.*

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