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A SIMPLE PROOF OF MIDY'S THEOREM

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Abstract: In this note we give a method for finding the elements in the period of a periodic rational number. We then use our method to give an elementary proof of Midy's theorem on repeating decimals.

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1. Introduction

It is well known that a rational number $x = \frac{m}{n}$ with $\gcd(m, n) = 1, n = 2^e\, 5^t\, b$, $\gcd(b, 10) = 1$, is periodic and its length of period is the order of 10 modulo b. Some periodic decimals has fascinating properties. For example, according to Dickson [1], E. Midy proved in 1836 that if the period of a reciprocal of a prime $p \geq 5$ has even length and is split into two half-periods then the sum of the halves is a string of 9's.

For example,

$$\frac{1}{7} = 0.\overline{142857}$$

with 142+857=999, and

$$\frac{1}{17} = 0.\overline{0588235294117647}$$

with 05882352+94117647=99999999.

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Midy's theorem holds also for reciprocals of some composite numbers.

For example,

$$\frac{1}{77} = 0.\overline{012987}$$

with $012 + 987 = 999$, and

$$\frac{1}{121} = 0.\overline{0082644628099173553719}$$

with $00826446280 + 99173553719 = 99999999999$.

2. Midy's Theorem

Many authors have given proofs of Midy's theorem: see, for example, [2],[3],[4], and [5]. In this paper we give a very simple proof of Midy's theorem based only on modular arithmetic.

We first give a simple method for finding the elements of the period of the reciprocal of a natural number m relatively prime to 10.

Lemma 1. *Let m be relatively prime to 10 and choose $0 \leq b_i < m; i = 1, 2, ..., r$ such that $10^i \equiv b_i \bmod m$, where r is the order of 10 modulo m. Then $\frac{1}{m} = 0.\overline{a_1 a_2 ... a_r}$, where a_i is congruent modulo 10 to:*

a) $9b_i$ *if* $m \equiv 1 \bmod 10$;

b) $3b_i$ *if* $m \equiv 3 \bmod 10$;

c) $7b_i$ *if* $m \equiv 7 \bmod 10$ *and*

d) b_i *if* $m \equiv 9 \bmod 10$.

Proof. Choose $h_i; i = 1, 2, ..., r$ such that $0 \leq h_i < m$, $10 = h_1 + ma_1$ and $10h_{i-1} = h_i + ma_i$ for $2 \leq i \leq r$. Then modulo m we have $h_1 \equiv 10$ and $h_i \equiv 10^{i-1}h_1 \equiv b_i$ for $i \geq 2$. Hence $h_i = b_i; i = 1, 2, ..., r$ and therefore $ma_i \equiv -b_i \bmod 10$.

Now if $m \equiv 1 \bmod 10$, then $a_i \equiv -b_i \equiv 9b_i \bmod 10$. If $m \equiv 3 \bmod 10$, then $3a_i \equiv -b_i$ so $a_i \equiv 3b_i \bmod 10$. If $m \equiv 7 \bmod 10$, then $7a_i \equiv -b_i$ so $a_i \equiv 7b_i \bmod 10$. Finally, if $m \equiv 9 \bmod 10$, then $9a_i \equiv -b_i$ so $a_i \equiv b_i \bmod 10$. □

Now we use our lemma to give a very simple proof of Midy's theorem.

Theorem 1. (see [3]) *Let m, b_i and r be as in Lemma 1. If $r = 2w$ is even and $10^w \equiv -1$ mod m, then*

$$\frac{1}{m} = 0.\overline{a_1 a_2 ... a_w a_{w+1} a_{w+2} ... a_{2w}}\,,$$

with $a_i + a_{w+i} = 9$; $i = 1, 2, ..., w$.

Proof. For $i = 1, 2, ..., r$ we have mod m, $b_{w+i} \equiv 10^{w+i} \equiv -10^i \equiv m - b_i$ so $b_{w+i} = m - b_i$. Now if $m \equiv 1$ mod 10, then by the above lemma $a_i + a_{w+i} \equiv 9b_i + 9(m - b_i) = 9m \equiv 9$. The three other cases are similar. □

Corollary 1. *If p is a prime and the period of $\frac{1}{p}$ has even length, say $r = 2w$, then*

$$\frac{1}{p} = 0.\overline{a_1 a_2 ... a_w a_{w+1} a_{w+2} ... a_{2w}}\,,$$

with $a_i + a_{w+i} = 9$; $i = 1, 2, ..., w$.

Proof. This follows immediately from the above theorem since if $r = 2w$ is the length of the period of $\frac{1}{p}$ then $10^w \equiv -1$ mod p. □

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