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SOME NEW DOUBLE SEQUENCE SPACES OVER n-NORMED SPACES

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Abstract: In this paper we introduce some new double sequence spaces defined by a sequence of Orlicz functions over n-normed spaces. We also study some topological properties and inclusion relation between these spaces.

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1. Introduction and Preliminaries

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [16] used the idea of Orlicz function to define the following sequence space

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\},$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with

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the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It was shown in [16] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$.

The initial work on double sequences can be found in Bromwich [4]. Later on they were studied by Hardy [13], Moricz [18], Moricz and Rhoades [19], Tripathy ([36], [37]), Başarır and Sonalcan [2] and many others. Hardy [13] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [39] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [23] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Next, Mursaleen [21] and Mursaleen and Edely [24] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{m,n})$ into one whose core is a subset of the M-core of x. By the convergence of a double sequence we mean the convergence in the Pringsheim sense, i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$, see [27]. We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l.

The concept of 2-normed spaces was initially developed by Gähler [8] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [20]. Since then, many others have studied this concept and obtained various results, see Gunawan ([10],[11]) and Gunawan and Mashadi [12] and references therein. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;

2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;

3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and

4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

The notion of difference sequence spaces was introduced by Kızmaz [14], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let s, v be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_v^s) = \{x = (x_k) \in w : (\Delta_v^s x_k) \in Z\},$$

where $\Delta_v^s x = (\Delta_v^s x_k) = (\Delta_v^{s-1} x_k - \Delta_v^{s-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^s x_k = \sum_{w=0}^{r} (-1)^w \begin{pmatrix} s \\ w \end{pmatrix} x_{k+vw}.$$

Taking $s = 1$, we get the spaces which were introduced and studied by Et and Çolak [7]. Taking $s = v = 1$, we get the spaces which were introduced and studied by Kızmaz [14].

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,

2. $p(-x) = p(x)$, for all $x \in X$,

3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,

4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda \ as \ n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0 \ as \ n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0 \ as \ n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [38], Theorem 10.4.2, P-183). For more details about sequence spaces (see [1], [3], [5], [6], [17], [22],[25], [26], [28], [29], [30], [31] [32]) and references therein.

A double sequence space E is said to be solid if $\alpha_{k,l}x_{k,l} \in E$ whenever $x_{k,l} \in E$ and for all double sequences $\alpha_{k,l}$ of scalars with $|\alpha_{k,l}| \leq 1$, for all $k, l \in \mathbb{N}$.

Let $\lambda = (\lambda_r)$ be a non-decreasing sequence of positive numbers tending to infinity and $\lambda_{r+1} \leq \lambda_r + 1, \ \ \lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{k \in I_r} x_k, \ \ I_r = [r - \lambda_r + 1, r].$$

A single sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $t_r(x) \to L$ as $r \to \infty$ see [12]. If $\lambda_r = r$, then the (V, λ)-summability is reduced to $(C, 1)$-summability see ([34, 35]).

The double sequence $\lambda_2 = (\lambda_{m,n})$ of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \leq \lambda_{m,n} + 1, \ \ \lambda_{m,n+1} \leq \lambda_{m,n} + 1,$$

$$\lambda_{m,n} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1}, \ \ \lambda_{1,1} = 1,$$

and

$$I_{m,n} = \Big\{(k, l) : m - \lambda_{m,n} + 1 \leq k \leq m, \ \ n - \lambda_{m,n} + 1 \leq l \leq n\Big\}.$$

The generalized double de Vallee-Poussin mean is defined by

$$t_{m,n} = t_{m,n}(x_{k,l}) = \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} x_{k,l}.$$

A double number sequence $x = (x_{k,l})$ is said to be (V_2, λ_2)-summable to a number L, if $P - \lim_{m,n} t_{m,n} = L$. If $\lambda_{m,n} = mn$, then the (V_2, λ_2)-summability is reduced to $(C, 1, 1)$-summability see [33]. We write

$$[V_2, \lambda_2] = \Big\{x = (x_{k,l}) : P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} |x_{k,l} - L| = 0, \ \ \text{for some} \ \ L\Big\}$$

for sets of double sequence $x = (x_{k,l})$. We say that $x = (x_{k,l})$ is strongly $[V_2, \lambda_2]$-summable to L, that is $x = (x_{k,l}) \to L([V_2, \lambda_2])$.

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a sequence of positive real numbers with $0 \leq p_{k,l} \leq \sup p_{k,l} = G$, $K = \max(1, 2^{G-1})$ then

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K\{|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\} \tag{1.1}$$

for all k, l and $a_{k,l}, b_{k,l} \in \mathbb{C}$. Also $|a|^{p_{k,l}} \leq \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers. In the present paper, we define the following sequence spaces:

$$\Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_0$$
$$= \Big\{x = (x_{k,l}) : P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}$$
$$= 0, \text{for some } \rho > 0\Big\},$$

$$\Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_1$$
$$= \Big\{x = (x_{k,l}) : P\text{-}\lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}$$
$$= 0, \text{for some } \rho > 0 \text{ and } L > 0\Big\}$$

and

$$\Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$$
$$= \Big\{x = (x_{k,l}) : \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}$$
$$< \infty, \text{ for some } \rho > 0\Big\}.$$

In the present paper we study some topological properties and inclusion relation between the above defined sequence spaces.

2. Some Topological Properties

Theorem 2.1. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers. Then the spaces $\Big[V_2, \lambda_2, \mathcal{M}, \Delta_s^r, u, p, \|\cdot, \cdots, \cdot\|\Big]_0$, $\Big[V_2, \lambda_2, \mathcal{M}, \Delta_s^r, u, p, \|\cdot, \cdots, \cdot\|\Big]_1$ and $\Big[V_2, \lambda_2, \mathcal{M}, \Delta_s^r, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x = (x_{k,l}), y = (y_{k,l}) \in \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1, ρ_2 such that

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} = 0,$$

$$\text{for some } \rho_1 > 0$$

and

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} = 0,$$

$$\text{for some } \rho_2 > 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M_{k,l}$ is non-decreasing, convex and so by using inequality (1.1), we have

$$\begin{aligned}
&\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_v^s (\alpha x_{k,l} + \beta y_{k,l})}{\rho_3}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
\leq &\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_v^s \alpha x_{k,l}}{\rho_3}, z_1, \cdots, z_{n-1}\|\Big) \\
&+ \Big(\|\frac{\Delta_v^s \beta y_{k,l}}{\rho_3}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
\leq &K \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \frac{1}{2^{p_{k,l}}} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&+ K \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \frac{1}{2^{p_{k,l}}} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_v^s y_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
\leq &K \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}
\end{aligned}$$

$$+ K \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s y_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$<\infty.$$

Thus $\alpha x + \beta y \in \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$. This proves that

$$\Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$$

is a linear space. Similarly, we can prove that $\Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_0$ and $\Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_1$ are linear spaces. □

Theorem 2.2. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers. Then the space $\Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$ is a paranormed space, paranormed by*

$$g(x) = \inf\Big\{(\rho)^{\frac{p_{k,l}}{H}} : \Big(\frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \leq 1\Big\},$$

where $0 < p_{k,l} \leq \sup p_{k,l} = G, H = \max(1, G)$.

Proof. (i) Clearly $g(x) \geq 0$ for $x = (x_{k,l}) \in \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$. Since $M_{k,l}(0) = 0$, we get $g(0) = 0$.

(ii) $g(-x) = g(x)$.

(iii) Let $x = (x_{k,l}), y = (y_{k,l}) \in \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$, then there exist positive numbers $\rho_1,\ \rho_2$ such that

$$\frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \leq 1,$$

and

$$\frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have
$u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s(x_{k,l}+y_{k,l})}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}$

$$\begin{aligned}
&= u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s(x_{k,l}+y_{k,l})}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&\leq u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}\|\Big) + \Big(\|\frac{\Delta_v^s y_{k,l}}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s y_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}
\end{aligned}$$

and thus

$$\begin{aligned}
g(x+y) = \inf\Big\{&(\rho_1+\rho_2)^{\frac{p_{k,l}}{H}} : \\
&\Big(\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s(x_{k,l}+y_{k,l})}{(\rho_1+\rho_2)}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \leq 1\Big\} \\
\leq \inf\Big\{&(\rho_1)^{\frac{p_{k,l}}{H}} : \\
&\Big(\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \leq 1\Big\} \\
&+ \inf\Big\{(\rho_2)^{\frac{p_{k,l}}{H}} : \\
&\Big(\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s y_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \leq 1\Big\}.
\end{aligned}$$

Now, let $\lambda \in \mathbb{C}$, then the continuity of the product follows from the following inequality:

$$\begin{aligned}
g(\lambda x) = \inf\Big\{&(\rho)^{\frac{p_{k,l}}{H}} : \\
&\Big(\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s \lambda x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \leq 1\Big\} \\
= \inf\Big\{&(|\lambda|s)^{\frac{p_{k,l}}{H}} : \\
&\Big(\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{s}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}} \leq 1\Big\},
\end{aligned}$$

where $s = \frac{\rho}{|\lambda|}$. This completes the proof of the theorem. $\square$

Theorem 2.3. *If $0 < p_{k,l} < q_{k,l} < \infty$ for each k and l, then*

$$\Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty \subseteq \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, q, \|\cdot, \cdots, \cdot\|\Big]_\infty.$$

Proof. Let $x = (x_{k,l}) \in \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$, then there exists some $\rho > 0$ such that

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} < \infty.$$

This implies that

$$u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} < 1,$$

for sufficiently large value of k and l. Since $M_{k,l}$ is non-decreasing, we get

$$\begin{aligned} &\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{q_{k,l}} \\ &\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\ &< \infty. \end{aligned}$$

Thus, $x = (x_{k,l}) \in \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, q, \|\cdot, \cdots, \cdot\|\Big]_\infty$. This completes the proof of the theorem. $\square$

Theorem 2.4. *(i) If $0 < \inf p_{k,l} \leq p_{k,l} < 1$, then*

$$\Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty \subset \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, \|\cdot, \cdots, \cdot\|\Big]_\infty.$$

(ii) If $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$, then

$$\Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, \|\cdot, \cdots, \cdot\|\Big]_\infty \subset \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty.$$

Proof. (i) Let $x = (x_{k,l}) \in \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$. Since $0 < \inf p_{k,l} \leq 1$, we have

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]$$

$$\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

and hence $x = (x_{k,l}) \in \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, \|\cdot, \cdots, \cdot\|\Big]_\infty$.

(ii) Let $p_{k,l}$ for each (k,l) and $\sup\limits_{k,l} p_{k,l} < \infty$. Let

$$x = (x_{k,l}) \in \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, \|\cdot, \cdots, \cdot\|\Big]_\infty.$$

Then, for each $0 < \epsilon < 1$, there exists a positive integer $\mathbb{N}$ such that

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big] \leq \epsilon < 1,$$

for all $m, n \in \mathbb{N}$. This implies that

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big].$$

Thus $x = (x_{k,l}) \in \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty$ and this completes the proof. □

Theorem 2.5. *For any sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ which satisfies Δ_2-condition, we have*

$$\Big[V_2, \lambda_2, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big] \subset \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big].$$

Proof. Let $x = (x_{k,l}) \in \Big[V_2, \lambda_2, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]$, then

$$A_{m,n} = P - \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} |\Delta_v^s x_{k,l} - L|^{p_{k,l}} \text{ for some } L.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_{k,l}(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $y_{k,l} = |\Delta_v^s x_{k,l} - L|$ and consider $\sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M_{k,l}(y_{k,l})\Big]^{p_{k,l}} = \sum_1 + \sum_2$ where the first summation is over $y_{k,l} \leq \delta$ and the second summation over $y_{k,l} > \delta$. Since $\mathcal{M} = (M_{k,l})$ is continuous. $\sum_1 < \epsilon$ and for $y_{k,l} > \delta$, we use the fact that $y_{k,l} < \frac{y_{k,l}}{\delta} < 1 + \frac{y_{k,l}}{\delta}$.
Since $\mathcal{M} = (M_{k,l})$ is non-decreasing and convex, it follows that

$$M_{k,l}(y_{k,l}) < M_{k,l}(1 + \frac{y_{k,l}}{\delta}) < \frac{1}{2}M_{k,l}(2) + \frac{1}{2}M_{k,l}(2\frac{y_{k,l}}{\delta}).$$

Since $\mathcal{M} = (M_{k,l})$ satisfies Δ_2-condition, therefore

$$M_{k,l}(y_{k,l}) < \frac{1}{2}K\frac{y_{k,l}}{\delta}M_{k,l}(2) + \frac{1}{2}K\frac{y_{k,l}}{\delta}M_{k,l}(2) = K\frac{y_{k,l}}{\delta}M_{k,l}(2).$$

Hence, $\sum_2 < \max(1, K\delta^{-1}M_{k,l}(2))^H A_{m,n}$, where $H = \sup_{k,l} p_{k,l}$. This proves that

$$\Big[V_2, \lambda_2, \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big] \subset \Big[V_2, \lambda_2, \mathcal{M}, \Delta_v^s, u, \|\cdot, \cdots, \cdot\|\Big]. \qquad \square$$

Theorem 2.6. *Let $\mathcal{M}' = (M'_{k,l})$ and $\mathcal{M}'' = (M''_{k,l})$ are sequences of Orlicz functions, then we have*

$$\Big[V_2, \lambda_2, \mathcal{M}', \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty \cap \Big[V_2, \lambda_2, \mathcal{M}'', \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty \subseteq \Big[V_2, \lambda_2, \mathcal{M}' + \mathcal{M}'', \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty.$$

Proof. Let

$$x = (x_{k,l}) \in \Big[V_2, \lambda_2, \mathcal{M}', \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty \bigcap \Big[V_2, \lambda_2, \mathcal{M}'', \Delta_v^s, u, p, \|\cdot, \cdots, \cdot\|\Big]_\infty.$$

Then

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} u_{k,l}\Big[M'_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} < \infty, \text{ for some } \rho_1 > 0$$

and

$$\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}u_{k,l}\Big[M''_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}<\infty,\ \text{for some}\ \ \rho_2>0.$$

Let $\rho=\max\{\rho_1,\rho_2\}$. The results follows from the inequality

$$\begin{aligned}&\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}u_{k,l}\Big[(M'_{k,l}+M''_{k,l})\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\\ =&\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}u_{k,l}\Big[M'_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\\ &+u_{k,l}\Big[M''_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\\ \le& K\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}u_{k,l}\Big[M'_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\\ &+K\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}u_{k,l}\Big[M''_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}.\end{aligned}$$

□

Theorem 2.7. *The sequence space* $\Big[V_2,\lambda_2,\mathcal{M},\Delta_v^s,u,p,\|\cdot,\cdots,\cdot\|\Big]_\infty$ *is solid.*

Proof. Let $x=(x_{k,l})\in\Big[V_2,\lambda_2,\mathcal{M},\Delta_v^s,u,p,\|\cdot,\cdots,\cdot\|\Big]_\infty$, i.e

$$\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}<\infty.$$

Let $(\alpha_{k,l})$ be double sequence of scalars such that $|\alpha_{k,l}|\le 1$ for all $k,l\in\mathbb{N}\times\mathbb{N}$. Then, we get

$$\begin{aligned}&\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s \alpha_{k,l}x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\\ &\qquad\le\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_v^s x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\end{aligned}$$

and this completes the proof. □

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