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A LIFTED HAAR BASIS

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Abstract: Starting from the first generation Haar wavelets, we construct second generation wavelets with higher vanishing moments. We show that these new wavelets form a Riesz basis for $L_2([0,1])$ and we compute the optimal Bessel bound.

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1. Introduction

An orthonormal basis $\{f_k\}_{k=1}^{\infty}$ of elements in $L_2([0,1])$ can be used to approximate every function $f \in L_2([0,1])$ by linear combination of the elements in $\{f_k\}_{k=1}^{\infty}$ (see [2, 4, 8]):

$$f = \sum_k a_k f_k, \quad a_k = \langle f, f_k \rangle .$$

This basis representation of a function f gives us characteristics of f by studying the coefficients a_k. One of the most well-known bases in $L_2([0,1])$ is the

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Fourier basis. Fourier proved that any continuous function can be represented as an infinite sum of sine and cosine functions. If the function is periodic, the Fourier basis representation will reveal frequencies by studying the coefficients. However, the Fourier basis is not always a good tool to reproduce a more general function in $L_2([0,1])$. Particularly, if the function is highly nonsmooth, the approximation using Fourier basis may not be a good choice [7].

A. Haar [3] introduced a new basis,

$$\left\{ \phi_{0,0},\ \psi_{j,m} : j \geq 0,\ m = \frac{k}{2^j},\ k = 1, \cdots, 2^{j+1} - 1 \right\},$$

where $\phi_{0,0} = \chi_{[0,1]}$ and $\psi_{j,m} = 2^j \left(\chi_{[\frac{m}{2} - \frac{1}{2^{j+1}}, \frac{m}{2}]} - \chi_{[\frac{m}{2}, \frac{m}{2} + \frac{1}{2^{j+1}}]} \right)$. This Haar basis is the simplest example of a multiresolution analysis (MRA). A traditional MRA of the space $L_2(\mathbb{R})$ or $L_2([0,1])$ consists of a sequence of nested subspaces that satisfies certain self-similarity relations, i.e., it consists of families of functions derived from a single function by dilation and translation, which together form a basis for $L_2(\mathbb{R})$ or $L_2([0,1])$. This type of functions are known as wavelets. Wavelets have been successfully used to represent functions in $L_2(\mathbb{R})$ or $L_2([0,1])$ because they provide a simple approach for dealing with local aspects of a function. We refer to such wavelets as first generation wavelets [2, 5, 8]. Recently, W. Sweldens introduced a more general version of an MRA where the wavelets are not necessarily dilates and translates of each other but still enjoy many of the powerful properties of first generation wavelets. These "wavelets" are referred to as second generation wavelets [6]. Second generation wavelets are useful in analyzing finite, non-periodic functions.

In this paper, we study the properties of a lifting operator that serves as a tool to generate second generation wavelets. Our objective is to construct a new basis for $L_2([0,1])$ with higher vanishing moments by applying a lifting operator to the Haar basis. We show that this new basis forms a Riesz basis for $L_2([0,1])$ and we compute the optimal Bessel bound. We note that Fourier transforms and series are involved in most of the techniques in first generation wavelet theory, however this new basis construction does not rely on Fourier transforms and series. Let us begin this introduction by defining the various notions that were just mentioned.

We say that a sequence $\{f_k\}_{k=1}^{\infty}$ in $L_2([0,1])$ is a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_k |\langle f, f_k \rangle|^2 \leq B \left\| f \right\|^2, \quad \forall f \in L_2([0,1]).$$

Any number B satisfying this equation is called a Bessel bound for $\{f_k\}_{k=1}^{\infty}$. A a Bessel sequence $\{f_k\}_{k=1}^{\infty}$ in $L_2([0,1])$ is a frame if there exists $A > 0$ such that

$$A\,\|f\|^2 \le \sum_k |\langle f, f_k\rangle|^2\,, \quad \forall f \in L_2([0,1]).$$

We say that a sequence $\{f_k\}_{k=1}^{\infty}$ in $L_2([0,1])$ is a Riesz basis if $span\,(\{f_k\}_{k=1}^{\infty})$ is complete in $L_2([0,1])$, and there are constants $A, B > 0$ such that for every finite scalar sequence $\{c_k\}_{k=1}^{\infty}$,

$$A \sum_k |c_k|^2 \le \left\| \sum_k c_k f_k \right\|^2 \le B \sum_k |c_k|^2.$$

It is easy to show that a Riesz basis is a frame and furthermore that any orthonormal basis is a Riesz basis. Initially we will be working with an orthonormal basis, and we show that after a lifting operator is applied, the new set forms a Riesz basis for $L_2([0,1])$.

2. Preliminaries

2.1. Multiresolution Analysis. We will not go into full discussion of properties of the second generation of MRAs and refer to the paper [6] for the details. Throughout the remainder of this paper we will refer to $L_2([0,1])$ as simply L_2, and, as we are studying the second generation case, we will generally omit the "second generation" adjective.

Definition 1 ([6]). A multiresolution analysis M of L_2 is a sequence of closed subspaces $M = \{V_j \subset L_2 : j \in \mathbb{N}_0\}$, so that:

1. $V_j \subset V_{j+1}$,
2. $\bigcup_{j\in J} V_j$ is dense in L_2,
3. for each $j \in \mathbb{N}_0$, V_j has a Riesz basis given by scaling functions $\{\phi_{j,k} : k \in K(j)\}$,

where $K(j)$ is an index set and $\mathbb{N}_0$ denotes the natural numbers including zero.

We note that (second generation) MRAs are more general than the traditional MRAs since the scaling functions are not required to be linear combinations of scaled versions of themselves.

In this paper, we study the Haar system whose first scaling function is defined as the the constant function

$$\phi_{0,0} = \chi_{[0,1]}.$$

As the level j increases, the scaling function is scaled down by a factor of $\frac{1}{2}$ and translated to form two new scaling functions. That is,

$$\phi_{1,0} = \chi_{[0,\frac{1}{2}]},\ \phi_{1,1} = \chi_{[\frac{1}{2},1]},$$

$$\phi_{2,0} = \chi_{[0,\frac{1}{4}]},\ \phi_{2,\frac{1}{2}} = \chi_{[\frac{1}{4},\frac{2}{4}]},\ \phi_{2,\frac{2}{2}} = \chi_{[\frac{2}{4},\frac{3}{4}]},\ \phi_{2,\frac{3}{2}} = \chi_{[\frac{3}{4},\frac{4}{4}]}, \dots,$$

$$\phi_{j,k} = \chi_{[\frac{k}{2},\frac{k}{2}+\frac{1}{2^j}]}, \dots,$$

where $k \in K(j) := \{0/2^{j-1}, 1/2^{j-1}, \dots, (2^j - 1)/2^{j-1}\}$. The Haar scaling functions are a simple example of a traditional MRA of L_2.

Given an MRA, there exists a dual MRA, $\{\widetilde{V}_i\}_{j\in\mathbb{N}_0}$, $\widetilde{V}_i \subset L_2$, with (dual) scaling functions $\widetilde{\phi}_{j,k'}$, $k' \in K(j)$, which are biorthogonal to the original scaling functions:

$$\left\langle \phi_{j,k}, \widetilde{\phi}_{j,k'} \right\rangle = \delta_{k,k'} \text{ for } k, k' \in K(j). \tag{1}$$

In the same way that scaling functions form a basis for V_j, wavelets form a basis for the orthogonal complement W_j where $V_j \oplus W_j = V_{j+1}$. The formal definition of a wavelet is the following:

Definition 2 ([6]). A set of functions $\{\psi_{j,m} : j \in \mathbb{N}_0, m \in M(j)\}$ is a set of wavelet functions if

1. the space $W_j = \text{close span}\{\psi_{j,m} : m \in M(j)\}$ is a complement of V_j in V_{j+1} and $W_j \perp \tilde{V}_j$, and
2. the set $\{\psi_{j,m}/\left\|\psi_{j,m}\right\| : j \in \mathbb{N}_0, m \in M(j)\} \cup \{\phi_{0,k}/\left\|\phi_{0,k}\right\| : k \in K(0)\}$ is a Riesz basis for L_2,

where $M(j)$ is an index set.

Using the definition of wavelets, V_{j+1} can be written as a summation of the wavelet functions and the zeroth-level scaling functions:

$$V_{j+1} = W_j + W_{j-1} + W_{j-2} ... + W_0 + V_0.$$

In the case of the Haar wavelets in L_2, we have $M(j) = K(j+1) \backslash K(j)$ and the Haar wavelet $\psi_{j,m}$ is

$$\psi_{j,m} = 2^j \left(\phi_{j+1,m-\frac{2}{2^{j+1}}} - \phi_{j+1,m} \right).$$

As the level j increases, the Haar wavelets are scaled and translated in the same way as the Haar scaling functions.

The dual MRA admits (dual) wavelet functions $\widetilde{\psi}_{j',m'}$, $m' \in M(j)$ which are biorthogonal to the original wavelets:

$$\left\langle \psi_{j,m}, \widetilde{\psi}_{j',m'} \right\rangle = \delta_{m,m'} \delta_{j,j'}.$$

The accuracy of approximation of an MRA is dependent on specific properties of the wavelets. One very important property we will focus on is vanishing moments. Wavelets are said to have N vanishing moments if they satisfy

$$\int_0^1 x^p \psi_{j,m}(x) dx = 0, \; p = 0, ..., N-1, \; \forall j, \; m. \tag{2}$$

Wavelets with a vanishing moment of one, such as the Haar wavelets, are capable of encoding the behavior of a 0-degree polynomial. Wavelets with a vanishing moment of two can represent a linear function. The N vanishing moment conditions provide wavelets with polynomial reproduction up to degree $N-1$.

2.2. The Lifting Operator. Lower-level primal and dual wavelets and scaling functions can be defined in terms of linear combinations of higher level scaling functions and their dual scaling functions, because $V_j, W_j \subset V_{j+1}$. The following are called refinement equations:

$$\phi_{j,k} = \sum_{l \in K(j+1)} h_{j,k,l} \phi_{j+1,l}, \quad \widetilde{\phi}_{j,k} = \sum_{l \in K(j+1)} \widetilde{h}_{j,k,l} \widetilde{\phi}_{j+1,l},$$

$$\psi_{j,m} = \sum_{l \in M(j+1)} g_{j,m,l} \phi_{j+1,l}, \quad \widetilde{\psi}_{j,m} = \sum_{l \in M(j+1)} \widetilde{g}_{j,m,l} \widetilde{\phi}_{j+1,l}.$$

We convert the above equations into filter operators in the following manner:

$$H_j = \begin{bmatrix} h_{j,k_0,l_0} & h_{j,k_0,l_1} & \cdots \\ h_{j,k_1,l_0} & h_{j,k_1,l_1} & \\ \vdots & & \ddots \end{bmatrix}, \quad \widetilde{H}_j = \begin{bmatrix} \widetilde{h}_{j,k_0,l_0} & \widetilde{h}_{j,k_0,l_1} & \cdots \\ \widetilde{h}_{j,k_1,l_0} & \widetilde{h}_{j,k_1,l_1} & \\ \vdots & & \ddots \end{bmatrix},$$

$$G_j = \begin{bmatrix} g_{j,m_0,l_0} & g_{j,m_0,l_1} & \cdots \\ g_{j,m_1,l_0} & g_{j,m_1,l_1} & \\ \vdots & & \ddots \end{bmatrix}, \quad \widetilde{G}_j = \begin{bmatrix} \widetilde{g}_{j,m_0,l_0} & \widetilde{g}_{j,m_0,l_1} & \cdots \\ \widetilde{g}_{j,m_1,l_0} & \widetilde{g}_{j,m_1,l_1} & \\ \vdots & & \ddots \end{bmatrix},$$

where $K(j) = \{k_0, k_1, ...\}$, $M(j) = \{m_0, m_1, ...\}$, and $K(j+1) = \{l_0, l_1, ...\}$. The operators take a set of scaling functions on a certain resolution level and map them to their corresponding set of wavelets or scaling functions on the previous resolution level. For example:

$$\left[\phi_{j,k}\right]_{k\in K(j)} = H_j \left[\phi_{j+1,l}\right]_{l\in K(j+1)}.$$

A set of filter operators is said to be biorthogonal if it satisfies the relations

$$\begin{bmatrix} \tilde{H}_j \\ \tilde{G}_j \end{bmatrix} \begin{bmatrix} H_j^* & G_j^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} H_j^* & G_j^* \end{bmatrix} \begin{bmatrix} \tilde{H}_j \\ \tilde{G}_j \end{bmatrix} = I. \tag{3}$$

The lifting scheme takes a simple set of biorthogonal filter operators (in our case the Haar system) and manipulates them to form a new set of biorthogonal filter operators with desirable properties.

Result 3. (see [6]) *Take an initial set of biorthogonal filter operators*

$$\{H_j, \tilde{H}_j, G_j, \tilde{G}_j\}.$$

Then a new set of biorthogonal filter operators $\{H_j^{new}, \tilde{H}_j^{new}, G_j^{new}, \tilde{G}_j^{new}\}$ *can be defined as*

$$\begin{aligned} H_j^{new} &= H_j, \\ \tilde{H}_j^{new} &= \tilde{H}_j + S_j\tilde{G}_j, \\ G_j^{new} &= G_j - S_j^* H_j, \\ \tilde{G}_j^{new} &= \tilde{G}_j, \end{aligned}$$

where S_j *is an operator form* $l^2(M(j))$ *to* $l^2(K(j))$*, called the lifting operator.*

It is unknown whether or not these new lifted filter operators correspond to a basis in L_2. In the following section, we construct a basis using a lift operator. The key to obtain this new basis is the third equation of Result 3, which reveals the effect of the lifting operator on wavelets:

$$\left[\psi_{j,m}^{new}\right]_{m\in M(j)} = \left[\psi_{j,m}\right]_{m\in M(j)} - S_j^* \left[\phi_{j,k}\right]_{k\in K(j)}. \tag{4}$$

3. A Lifted Haar Basis

This section focuses on finding a lifting operator such that the lifted scheme applied to the Haar wavelets produces new wavelets with increased vanishing moments. From now on when we use ϕ or ψ we will be referring to the Haar scaling functions and wavelets. The Haar wavelets originally have a vanishing moment of one. Using Equation (4), we attempt to find S_j^* such that the new lifted wavelets have a vanishing moment of two:

$$\int_0^1 \psi_{j,m}^{new}(x)dx = 0 \text{ and } \int_0^1 x\ \psi_{j,m}^{new}(x)dx = 0, \text{ for any } j, m.$$

From Equation (4) we have in matrix notation:

$$\left[\int_0^1 \psi_{j,m}(x)\,dx, \int_0^1 x\ \psi_{j,m}(x)\,dx\right]_m = S_j^* \left[\int_0^1 \phi_{j,k}(x)\,dx, \int_0^1 x\ \phi_{j,k}(x)\,dx\right]_k,$$

where $m \in M(j)$ and $k \in K(j)$.

There is no solution for S_0^*. There is a unique solution $S_1^* = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$.

For $j \geq 2$, we choose a solution of the following form:

$$S_j^* = 2^{j-1} \begin{bmatrix} S_1^* & 0 & 0 & 0 & \cdots \\ 0 & S_1^* & 0 & 0 & \cdots \\ 0 & 0 & S_1^* & 0 & \cdots \\ 0 & 0 & 0 & S_1^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = 2^{j-2} \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which is a 2^j x 2^j matrix. We denote the elements of $M(j-1)$ by

$$k_1, k_2, \ldots, k_{2^{j-1}},$$

and the elements of $M(j)$ and $K(j)$ by

$$k_1 - \frac{1}{2^j}, k_1 + \frac{1}{2^j}, k_2 - \frac{1}{2^j}, k_2 + \frac{1}{2^j}, \ldots, k_{2^{j-1}} - \frac{1}{2^j}, k_{2^{j-1}} + \frac{1}{2^j}, \text{ and}$$

$$k_1 - \frac{2}{2^j}, k_1, k_2 - \frac{2}{2^j}, k_2, \ldots, k_{2^{j-1}} - \frac{2}{2^j}, k_{2^{j-1}},$$

respectively. For each $j \geq 1$, from Equation (4), we obtain the following:

$$\begin{aligned} \psi_{j,k-\frac{1}{2^j}}^{new} &= \psi_{j,k-\frac{1}{2^j}} - 2^{j-2}\left(\phi_{j,k-\frac{2}{2^j}} - \phi_{j,k}\right), \\ \psi_{j,k+\frac{1}{2^j}}^{new} &= \psi_{j,k+\frac{1}{2^j}} - 2^{j-2}\left(\phi_{j,k-\frac{2}{2^j}} - \phi_{j,k}\right), \end{aligned} \tag{5}$$

where $k = k_1, ..., k_{2^{j-1}}$.

Our goal in this section is to show that the normalized level one scaling functions and the normalized new functions $\overline{\psi_{j,m}^{new}}$ for $j \geq 1$ form a Riesz basis for L_2, where normalization is defined by $\overline{f} = \frac{f}{\|f\|}$ for $\|f\| \neq 0$. The level one scaling functions are required because the lifting operator has no solution on resolution level zero.

Theorem 4. *The lifted Haar system* $\left\{ \overline{\phi_{1,0}}, \overline{\phi_{1,1}}, \overline{\psi_{j,m}^{new}} \,:\, j \geq 1,\, m \in M(j) \right\}$ *is a Riesz basis of* $L_2\left([0,1]\right)$.

The proof of this theorem follows from the perturbation result:

Result 5. (see [1]) *Let* $\{f_k\}_{k=1}^{\infty}$ *be a frame for* $\mathcal{H}$ *with bounds* A, B. *Let* $\{g_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{H}$ *and assume that there exist constants* $\lambda, \mu \geq 0$ *such that* $\lambda + \frac{\mu}{\sqrt{A}} < 1$ *and*

$$\left\| \sum_k c_k (f_k - g_k) \right\| \leq \lambda \left\| \sum_k c_k f_k \right\| + \mu \left(\sum_k |c_k|^2 \right)^{1/2},$$

for all finite scalar sequences c_k. *Then* $\{g_k\}_{k=1}^{\infty}$ *is a frame for* $\mathcal{H}$ *with bounds*

$$A\left(1 - \left(\lambda + \frac{\mu}{\sqrt{A}}\right)\right)^2, B\left(1 + \lambda + \frac{\mu}{\sqrt{B}}\right)^2.$$

Moreover, if $\{f_k\}_{k=1}^{\infty}$ *is a Riesz basis, then* $\{g_k\}_{k=1}^{\infty}$ *is a Riesz basis.*

We note that by the definition of the Haar system, we have

$$\psi_{j,k\pm\frac{1}{2^j}}^{new} = \psi_{j,k\pm\frac{1}{2^j}} - \frac{1}{2}\psi_{j-1,k},$$

which implies

$$\begin{aligned} & \left\langle \psi_{j,k\pm\frac{1}{2^j}}^{new}, \psi_{j',k'\pm\frac{1}{2^{j'}}}^{new} \right\rangle \qquad (6) \\ = & \left\langle \psi_{j,k\pm\frac{1}{2^j}}, \psi_{j',k'\pm\frac{1}{2^{j'}}} \right\rangle + \left\langle -\frac{1}{2}\psi_{j-1,k}, \psi_{j',k'\pm\frac{1}{2^{j'}}} \right\rangle \\ + & \left\langle \psi_{j,k\pm\frac{1}{2^j}}, -\frac{1}{2}\psi_{j'-1,k'} \right\rangle + \left\langle -\frac{1}{2}\psi_{j-1,k}, -\frac{1}{2}\psi_{j'-1,k'} \right\rangle. \end{aligned}$$

Proof of Theorem 4. Due to the biorthogonality of the Haar basis, from Equation (6), we get $\left\|\psi^{new}_{j,k\pm\frac{1}{2^j}}\right\| = 3\sqrt{2^{j-3}}$. We therefore have

$$\overline{\psi^{new}_{j,k\pm\frac{1}{2^j}}} = \frac{2\sqrt{2}}{3}\overline{\psi_{j,k-\frac{1}{2^j}}} - \frac{1}{3}\overline{\psi_{j-1,k}}.$$

Let $\{\, C_{1,0}, C_{1,1}, C_{j,m} : j \geq 1, m \in M(j)\,\}$ be a square-summable sequence and let

$$D_1 = C_{1,0}\overline{\phi_{1,0}} + C_{1,1}\overline{\phi_{1,1}} + \sum_{j\geq 1, m\in M(j)} C_{j,m}\overline{\psi_{j,m}},$$
$$D_2 = C_{1,0}\overline{\phi_{1,0}} + C_{1,1}\overline{\phi_{1,1}} + \sum_{j\geq 1, m\in M(j)} C_{j,m}\overline{\psi^{new}_{j,m}}.$$

We want to show that

$$\|D_1 - D_2\| \leq \mu \left(|C_{1,0}|^2 + |C_{1,1}|^2 + \sum_{j\geq 1, m\in M(j)} |C_{j,m}|^2 \right)^{\frac{1}{2}}, \tag{7}$$

for some $\mu < 1$, which implies that

$$\left\{\overline{\phi_{1,0}}, \overline{\phi_{1,1}}, \overline{\psi^{new}_{j,m}} : j \geq 1, m \in M(j)\right\}$$

is a Riesz basis by Result 5. We have

$$\|D_1 - D_2\| = \left\| \sum_{j\geq 1, m\in M(j)} \left(\overline{\psi_{j,m}} - \overline{\psi^{new}_{j,m}}\right) \right\|. \tag{8}$$

Following the notation of Equations (5), we have

$$\sum_{j\geq 1, m\in M(j)} \overline{\psi_{j,m}} = \sum_{j\geq 1, k\in M(j-1)} \left(\overline{\psi_{j,k-\frac{1}{2^j}}} + \overline{\psi_{j,k+\frac{1}{2^j}}}\right),$$

and we can rewrite the right-hand side of Equation (8) as

$$\begin{aligned}
(8) \quad &= \Big\| \textstyle\sum_{j\geq 1, k\in M(j-1)} \Big(C_{j,k-\frac{1}{2^j}} \Big[\overline{\psi_{j,k-\frac{1}{2^j}}} - \Big(\frac{2\sqrt{2}}{3}\overline{\psi}_{j,k-\frac{1}{2^j}} - \frac{1}{3}\overline{\psi_{j-1,k}} \Big) \Big] \\
&\qquad + C_{j,k+\frac{1}{2^j}} \Big[\overline{\psi_{j,k+\frac{1}{2^j}}} - \Big(\tfrac{2\sqrt{2}}{3}\overline{\psi_{j,k+\frac{1}{2^j}}} - \tfrac{1}{3}\overline{\psi_{j-1,k}} \Big) \Big] \Big) \Big\| \\
&= \Big\| \Big(1 - \tfrac{2\sqrt{2}}{3}\Big) \textstyle\sum_{j\geq 1, k\in M(j-1)} \Big(C_{j,k-\frac{1}{2^j}}\overline{\psi_{j,k-\frac{1}{2^j}}} + C_{j,k+\frac{1}{2^j}}\overline{\psi_{j,k+\frac{1}{2^j}}} \Big) \\
&+ \tfrac{1}{3} \textstyle\sum_{j\geq 1, k\in M(j-1)} C_{j,k-\frac{1}{2^j}}\overline{\psi_{j-1,k}} + \tfrac{1}{3}\sum_{j\geq 1, k\in M(j-1)} C_{j,k+\frac{1}{2^j}}\overline{\psi_{j-1,k}} \Big\|.
\end{aligned}$$

By the triangle inequality we have

$$\begin{aligned}(8)\qquad &\leq \left(1-\tfrac{2\sqrt{2}}{3}\right)\left\|\textstyle\sum_{j\geq 1,k\in M(j-1)}\left(C_{j,k-\frac{1}{2^j}}\overline{\psi_{j,k-\frac{1}{2^j}}}+C_{j,k+\frac{1}{2^j}}\overline{\psi_{j,k+\frac{1}{2^j}}}\right)\right\|\\ &+\tfrac{1}{3}\left\|\textstyle\sum_{j\geq 1,k\in M(j-1)}C_{j,k-\frac{1}{2^j}}\overline{\psi_{j-1,k}}\right\|+\tfrac{1}{3}\left\|\textstyle\sum_{j\geq 1,k\in M(j-1)}C_{j,k+\frac{1}{2^j}}\overline{\psi_{j-1,k}}\right\|.\end{aligned}$$

By the definition of a Riesz basis and the orthogonality of the Haar baisis, we have that $\left\|\sum\limits_{j\geq 1,k\in M(j-1)}\left(C_{j,k-\frac{1}{2^j}}\overline{\psi_{j,k-\frac{1}{2^j}}}+C_{j,k+\frac{1}{2^j}}\overline{\psi_{j,k+\frac{1}{2^j}}}\right)\right\|$, $\left\|\sum\limits_{j\geq 1,k\in M(j-1)}C_{j,k-\frac{1}{2^j}}\overline{\psi_{j-1,k}}\right\|$ and $\left\|\sum\limits_{j\geq 1,k\in M(j-1)}C_{j,k+\frac{1}{2^j}}\overline{\psi_{j-1,k}}\right\|$ are each less than or equal to

$$\left(|C_{1,0}|^2+|C_{1,1}|^2+\sum_{j\geq 1,m\in M(j)}\left(|C_{j,m}|^2\right)\right)^{\frac{1}{2}}.$$

Therefore,

$$\|D_1-D_2\|\leq\frac{5-2\sqrt{2}}{3}\left(|C_{1,0}|^2+|C_{1,1}|^2+\sum_{j\geq 1,m\in M(j)}|C_{j,m}|^2\right)^{\frac{1}{2}},$$

which completes the proof. ☐

Using the biorthogonality of a new set of filter operators

$$\{H_j^{new},\tilde{H}_j^{new},G_j^{new},\tilde{G}_j^{new}\}$$

and refinement equations from Section 2.2, the set

$$\left\{\overline{\widetilde{\phi}_{1,0}},\overline{\widetilde{\phi}_{1,1}},\overline{\widetilde{\psi}_{j,m}^{new}}\;:\;j\geq 1,\,m\in M(j)\right\}$$

is biorthogonal to $\left\{\overline{\phi_{1,0}},\overline{\phi_{1,1}},\overline{\psi_{j,m}^{new}}\;:\;j\geq 1,\,m\in M(j)\right\}$.

Corollary 6. *The set*

$$\left\{\overline{\widetilde{\phi}_{1,0}},\overline{\widetilde{\phi}_{1,1}},\overline{\widetilde{\psi}_{j,m}^{new}}\;:\;j\geq 1,\,m\in M(j)\right\}$$

is the dual Riesz basis of the lifted Haar basis.

We note that $\mu=\frac{5-2\sqrt{2}}{3}$ provides us with an upper bound, but it is not the optimal upper bound. In the following section, we calculate the optimal upper bound of this new Riesz basis by calculating the optimal Bessel bound.

4. The Optimal Upper Bound for the Lifted Haar Basis

The condition on a Bessel sequence is more briefly summarized in a result by Christensen:

Result 7. (see [2]) *Let* $\{f_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{H}$ *and assume that there exists a constant* $B > 0$ *such that*

$$\sum_{k=1}^{\infty} |\langle f_j, f_k \rangle|^2 \leq B, \qquad \textit{for each } j.$$

Then $\{f_k\}_{k=1}^{\infty}$ *is a Bessel sequence with bound* B*.*

To obtain the best Bessel bound, we calculate

$$\left\langle \psi^{new}_{j,k\pm\frac{1}{2^j}}, \psi^{new}_{j',k'\pm\frac{1}{2^{j'}}} \right\rangle, \text{ and } \left\langle \phi_{1,l}, \psi^{new}_{j,m} \right\rangle,$$

for all j, j', k, k', and $l = 0, 1$. The calculations are contained in the next five propositions. Each proposition is proved using Equation (6) and the biorthogonality of the Haar basis. We state the results without giving detailed computations.

Proposition 8. *For each* $j \geq 2$*,* $k \in M(j-2)$*, and* $m, m' \in M(j)$*, we define* $k_1 := k + \frac{1}{2^{j-1}}$ *and* $k_2 := k - \frac{1}{2^{j-1}}$*. Then*

$$\left\langle \psi^{new}_{j,m}, \psi^{new}_{j-1,m'} \right\rangle = \begin{cases} -2^{j-2}, & \textit{if } m = k_1 \pm \frac{1}{2^j} \textit{ and } m' = k_1, \\ -2^{j-2}, & \textit{if } m = k_2 \pm \frac{1}{2^j} \textit{ and } m' = k_2, \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition 9. *Let* $k \in M(j-1)$ *for* $j \geq 1$*. Then*

$$\left\langle \psi^{new}_{j,m}, \psi^{new}_{j,m'} \right\rangle = \begin{cases} 2^j + 2^{j-3}, & \textit{if } m = m', \\ 2^{j-3}, & \textit{if } m = k + \frac{1}{2^j} \textit{ and } m' = k - \frac{1}{2^j}, \\ 0, & \textit{otherwise.} \end{cases}$$

Proposition 10. *For each* $j \geq 1$ *and* $k \in M(j-1)$*, we define* $k_1 := k + \frac{1}{2^{j-1}}$

and $k_2 := k - \frac{1}{2^{j-1}}$. *Then*

$$\left\langle \psi^{new}_{j,m}, \psi^{new}_{j+1,m'} \right\rangle = \begin{cases} -2^{j-1}, & \text{if } m = k_1 \text{ and } m' = k_1 \pm \frac{1}{2^{j+1}}, \\ -2^{j-1}, & \text{if } m = k_2 \text{ and } m' = k_2 \pm \frac{1}{2^{j+1}}, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 11. *For each* $j \geq 1$,

$$\left\langle \psi^{new}_{j,m}, \psi^{new}_{j+n,m'} \right\rangle = 0,$$

where $m \in M(j), |n| \geq 2, j+n \geq 0$, *and* $m' \in M(j+n)$.

Proposition 12. *For each* $j \geq 2$ *and for any* $m \in M(j)$,

$$\left\langle \phi_{1,0}, \psi^{new}_{j,m} \right\rangle = \left\langle \phi_{1,1}, \psi^{new}_{j,m} \right\rangle = 0.$$

Furthermore, we have

$$\left\langle \phi_{1,0}, \psi^{new}_{1,\frac{1}{2}} \right\rangle = \left\langle \phi_{1,0}, \psi^{new}_{1,\frac{3}{2}} \right\rangle = \frac{-1}{4}, \ \left\langle \phi_{1,1}, \psi^{new}_{1,\frac{1}{2}} \right\rangle = \left\langle \phi_{1,1}, \psi^{new}_{1,\frac{3}{2}} \right\rangle = \frac{1}{4}.$$

Finally, after normalization and summing, we obtain the following:

$$\left|\left\langle \overline{\phi_{1,0}}, \overline{\phi_{1,0}} \right\rangle\right|^2 + \left|\left\langle \overline{\phi_{1,0}}, \overline{\phi_{1,1}} \right\rangle\right|^2 + \sum_{j\geq 1, m\in M(j)} \left|\left\langle \overline{\phi_{1,0}}, \overline{\psi^{new}_{j,m}} \right\rangle\right|^2 = \frac{10}{9},$$

$$\left|\left\langle \overline{\phi_{1,1}}, \overline{\phi_{1,0}} \right\rangle\right|^2 + \left|\left\langle \overline{\phi_{1,1}}, \overline{\phi_{1,1}} \right\rangle\right|^2 + \sum_{j\geq 1, m\in M(j)} \left|\left\langle \overline{\phi_{1,1}}, \overline{\psi^{new}_{j,m}} \right\rangle\right|^2 = \frac{10}{9},$$

$$\left|\left\langle \overline{\psi_{1,\frac{1}{2}}}, \overline{\phi_{1,0}} \right\rangle\right|^2 + \left|\left\langle \overline{\psi_{1,\frac{1}{2}}}, \overline{\phi_{1,1}} \right\rangle\right|^2 + \sum_{j\geq 1, m\in M(j)} \left|\left\langle \overline{\psi_{1,\frac{1}{2}}}, \overline{\psi^{new}_{j,m}} \right\rangle\right|^2 = \frac{107}{81},$$

$$\left|\left\langle \overline{\psi_{1,\frac{3}{2}}}, \overline{\phi_{1,0}} \right\rangle\right|^2 + \left|\left\langle \overline{\psi_{1,\frac{3}{2}}}, \overline{\phi_{1,1}} \right\rangle\right|^2 + \sum_{j\geq 1, m\in M(j)} \left|\left\langle \overline{\psi_{1,\frac{3}{2}}}, \overline{\psi^{new}_{j,m}} \right\rangle\right|^2 = \frac{107}{81},$$

and for $j' \geq 2, m' \in M(j')$,

$$\left|\left\langle \overline{\psi_{j',m'}}, \overline{\phi_{1,0}} \right\rangle\right|^2 + \left|\left\langle \overline{\psi_{j',m'}}, \overline{\phi_{1,1}} \right\rangle\right|^2 + \sum_{j\geq 1, m\in M(j)} \left|\left\langle \overline{\psi_{j',m'}}, \overline{\psi^{new}_{j,m}} \right\rangle\right|^2 = \frac{106}{81}.$$

Theorem 13. *The optimal Bessel bound for the Riesz basis*

$$\left\{\overline{\phi_{1,0}}, \overline{\phi_{1,1}}, \overline{\psi_{j,m}^{new}} : j \geq 1, m \in M(j)\right\} \quad \textit{is} \quad \frac{107}{81}.$$

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