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OSCILLATION CRITERIA FOR A CLASS OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS WITH DAMPING TERM

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Abstract: Sufficient conditions for the oscillation of second order nonlinear damped difference equation are obtained. An example is given to illustrate these results.

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1. Introduction

In this paper we consider the oscillation behaviour of solution of second order nonlinear difference equations of the form

$$\Delta\Big(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)\Big) + p(n)|\Delta y(n)|^{\alpha-1}\Delta y(n) + q(n)f(y(n)) = 0, \quad (1)$$

where $r(n) \in \mathbb{C}^1\Big([t_0,\infty);\mathbb{R}^+\Big), p(n), q(n)$ are real sequences, α is the positive constant and f is a continuous real valued function of the real line $\mathbb{R}$ and satisfies $yf(y) \geq 0$ for $y \neq 0$. We restricted our attention to a solution $y(n)$ of (1) which exists on same half line $[n \leq n_0 < \infty)$.

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A solution $y(n)$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. Here, Δ is the forward difference operator defined by

$$\Delta x_n = x_{n+1} - x_n,$$

see [1]. In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation or non oscillation of the solution for difference equations and their analogous differential equations.

The paper is arranged as follows. In Section 2 we eastblish our main results. Finally, an example is given to illustrate our results.

2. Main Results

Lemma 2.1. *(i) $M \in [J \times J \times R, R]$ $M(n, j, u)$ is monotone non decreasing in u for each fixed (n, j) and one of the inequalities*

$$u(n) \leq h(n) + \sum_{j=n_0}^{k} M(n, j, u(j)); \;\; v(n) \geq h(n) + \sum_{j=n_0}^{k} M(n, j, v(j)) \tag{2}$$

is strict where $u, v, h \in C[J, R]$, J being a half line $[0, \infty)$.

(ii) $u(n_0) < v(n_0)$, then we have

$$u(n) < v(n). \tag{3}$$

Proof. Assuming the contrary, there exists n_1 such that

$$u(n_1) = v(n_1) \quad n_0 \leq n < n_1, \tag{4}$$

since n is monotone nondecreasing in u

$$M(n_1, j, u(j)) \leq M(n_1, j, v(j)),$$

(2) becomes

$$\begin{aligned} u(n_1) &\leq h(n_1) + \sum_{j=n_0}^{n_1} M(n_1, j, u(j)) \\ &\leq h(n_1) + \sum_{j=n_0}^{n_1} M(n_1, j, v(j)) \leq v(n_1). \end{aligned}$$

This a contradiction to that fact $u(n_1) = v(n_1)$. Hence (3) is true. $\square$

Lemma 2.2. *Let $G \in C[J \times J \times R_+, R]$, $G(n, j, u(j))$ be a monotone nondecreasing in u for each (n, j) and*

$$m(n) \leq m(n_0) + \sum_{j=n_0}^{n} G(n, j, m(j)),\ n \geq n_0,$$

where $m \in C[J, R_+]$ suppose that $r(n)$ is the maximal solution of the equation

$$y(n) = u_0(n_0) + \sum_{j=n_0}^{k} G(k, j, u(j)) \tag{5}$$

existing on J. Then the inequality $m(k_0) \leq u_0(k_0)$ implies that,

$$m(k) \leq r(n),\ k \geq k_0. \tag{6}$$

Proof. Let $u(k, \epsilon)$ be any solution of the equation

$$u(k, \epsilon) = u_0(k) + \epsilon + \sum_{k_0}^{k} G(k, j, u(j))$$

for $\epsilon > 0$ sufficiently small. Since $\lim\limits_{\epsilon \to 0} u(k, \epsilon) = r(n)$. It is enough to show that $m(k) < u(k, \epsilon)$

$$u(k, \epsilon) > u_0(k) + \sum_{j=k_0}^{k} G(k, j, u(j)). \tag{7}$$

Hence by (2) of Lemma 2.1, (6) is valid. □

Theorem 2.3. *Assume that $\Delta f(y) \geq 0; p(n) \leq 0; q(n) > 0$ and $\sum\limits_{j=n_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(j)} = \infty$ hold. Suppose that there exists a positive function $\rho(n)$ such that*

$$\sum_{j=n_0}^{\infty} q(j)\rho(j) = \infty, \tag{8}$$

$$p(n)\rho(n) \geq r(n)\Delta\rho(n). \tag{9}$$

Then every solution of (1) is oscillatory.

Proof. Assume that (1) has a non oscillatory solution. Without loss of generality suppose that it is an eventually positive solution (if it is eventually negative solution the proof is similar) that is $y(n) > 0$ for all $n > n_0$.

Suppose that $\Delta y(n)$ is oscillatory. It is enough to prove that $\Delta y(n)\Delta y(n+1) \leq 0$.

Case 1: Suppose that $\Delta y(n) < 0$ from (1), we obtain

$$\begin{aligned}
\Delta(r(n)(-\Delta y(n)^{\alpha-1}\Delta y(n))) &= -p(n)(-\Delta y(n)^{\alpha-1}\Delta y(n)) - q(n)f(y(n)) \\
-\Delta(r(n)(-\Delta y(n)^{\alpha-1}\Delta y(n))) &= p(n)(-\Delta y(n)^{\alpha-1}\Delta y(n)) - q(n)f(y(n)) \\
\Delta(r(n)(-\Delta y(n)^{\alpha})) &= -p(n)(-\Delta y(n)^{\alpha}) - q(n)f(y(n)) \\
\Delta(r(n)(-\Delta y(n)^{\alpha})) &= p(n)(-\Delta y(n)^{\alpha}) - q(n)f(y(n)) \\
\Delta(r(n)(-\Delta y(n)^{\alpha})) &= -p(n)(-\Delta y(n)^{\alpha}) + q(n)f(y(n)). \qquad (10)
\end{aligned}$$

Then there exists a $M > 0$ and $n_1 \geq n_0$ such that

$$(r(n)(-\Delta y(n)^{\alpha})) \geq M \qquad (11)$$

$$y(n) \leq -M^{\frac{1}{\alpha}} \frac{1}{\sum\limits_{j=n_1}^{\infty} (r(j))^{\frac{1}{\alpha}}}$$

which means that $\lim\limits_{n\to\infty}$ this contradicts assumption that $y(n) > 0$.

Case 2: Suppose that $\Delta y(n) > 0$ and let $\omega(n) = \rho(n)r(n)[\Delta y(n)]^{\alpha}$. Then

$$\Delta\omega(n) = E((\Delta y(n))^{\alpha} r(n))\Delta\rho(n) + \rho(n)\Delta r(n)(\Delta^2 y(n))^{\alpha}. \qquad (12)$$

Since $\Delta y(n) > 0$ then equation (1) becomes,

$$\Delta(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)) + p(n)|\Delta y(n)|^{\alpha-1}\Delta y(n) + q(n)f(y(n)) = 0$$

$$\Delta r(n)(\Delta^2 y(n))^{\alpha} + p(n)(\Delta y(n))^{\alpha} + q(n)f(y(n)) = 0.$$

Now (12) becomes,

$$\begin{aligned}
\frac{\Delta\omega(n)}{f(y(n))} &= \frac{E((\Delta y(n))^{\alpha} r(n)\Delta\rho(n))}{f(y(n))} + \frac{\rho(n)}{f(y(n))}(-p(n)(\Delta y(n))^{\alpha}) - q(n)f(y(n)) \\
&= \frac{E(r(n)(\Delta y(n))^{\alpha})\Delta\rho(n)}{f(y(n))} - \frac{\rho(n)p(n)(\Delta y(n))^{\alpha}}{f(y(n))} - \frac{\rho(n)q(n)f(y(n))}{f(y(n))} \\
&= -\rho(n)q(n) - \frac{\rho(n)p(n)(\Delta y(n))^{\alpha}}{f(y(n))} - \frac{\Delta\rho(k)E(r(n)(\Delta y(n)))}{f(y(n))}.
\end{aligned}$$

Consider,

$$\Delta(\frac{\omega(n)}{f(y(n))}) = \frac{f(y(n))\Delta\omega(n) - \omega(n)\Delta f(y(n))}{f(y(n))f(y(n+1))}$$

$$= \frac{f(y(n))}{f(y(n+1))}\Big(-\rho(n)q(n) - \frac{\rho(n)p(n)(\Delta y(n))^{\alpha}}{f(y(n))} +$$

$$\frac{r(n+1)(\Delta(y(n+1)))^{\alpha}\Delta\rho(n)}{f(y(n))}\Big) - \frac{\omega(n)\Delta f(y(n))}{f(y(n))f(y(n+1))}.$$

Applying summation on both the sides

$$\Delta(\frac{\omega(n)}{f(y(n))}) = \frac{\omega(n+1)}{f(y(n+1))} - \frac{\omega(n)}{f(y(n))}$$

$$= \sum_{j=n_0}^{n-1}(\frac{\omega(n+1)}{f(y(n+1))} - \frac{\omega(n)}{f(y(n))})$$

$$= \frac{\omega(n_0+1)}{f(u(n_0+1))} - \frac{\omega(n_0)}{f(y(n_0))}$$

$$+ \frac{\omega(n_0+2)}{f(u(n_0+2))} - \frac{\omega(n_0+1)}{f(y(n_0+1))}$$

$$\vdots$$

$$+ \frac{\omega(n)}{f(y(n))} - \frac{\omega(n-1)}{f(y(n-1))}$$

$$= \frac{\omega(n)}{f(y(n))} - \frac{\omega(n_0)}{f(y(n_0))}.$$

Equating the L.H.S. and R.H.S

$$\frac{\omega(n)}{f(y(n))} - \frac{\omega(n_0)}{f(y(n_0))} = \sum_{j=n_0}^{n-1}(-\frac{\rho(j)q(j)f(y(j))}{f(y(j+1))} - \frac{\rho(j)p(j)(\Delta y(j))}{f(y(j+1))})$$

$$+ \quad \frac{r(j+1)\Delta(y(j+1))\Delta\rho(j)}{f(y(j+1))} - \frac{\omega(n)\Delta f(y(j))}{f(y(j))f(y(j+1))}$$

as $n \to \infty$

$$0 < \lim_{n\to\infty}\frac{\omega(n)}{f(y(n))} \leq -\infty.$$

This is a contradiction. Hence $y(n)$ is oscillatory. □

Remark 2.4. If we replace $p(n) \leq 0, q(n) > 0$ by $p(n) \leq 0, q(n) \leq 0$, $\lim\limits_{n\to\infty} \frac{p(n)}{q(n)} = M > 0$, Theorem 2.3 also holds.

Theorem 2.5. *Assume that $\Delta f(u) \geq 0$ hold. Suppose also that*

$$\rho_0(n) = \sum_{j=n_0}^{n} \frac{\rho(j)p(j-1)}{r(j-1)} \tag{13}$$

$$\sum_{j=n_0}^{\infty} \frac{1}{[\rho_0(n)r(n)]^{\frac{1}{\alpha}}} = \infty \tag{14}$$

and $\rho_0(n)$ s.t. (8) holds. Then every solution of (1) is oscillatory.

Proof. To the contrary (1) has nonoscillatory solution $y(n)$. Without loss of generality, we assume that $y(n)$ is an eventually positive solution. Let

$$\omega(n) = \rho_0(n)r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)$$

and

$$\Delta\omega(n) = \rho_0(n+1)\Delta(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n))$$

$$+r(n)(|\Delta y(n)|^{\alpha-1}\Delta y(n)\Delta\rho_0(k)).$$

From (13), we obtain

$$\rho_0(n) = \sum_{j=n_0}^{k} \frac{\rho_0(j)p(j-1)}{r(j-1)}$$

$$\Delta\rho_0(n) = \sum_{j=n_0}^{n+1} \frac{\rho_0(j)p(j-1)}{r(j-1)} - \sum_{j=n_0}^{n} \frac{\rho_0(j)p(j-1)}{r(j-1)}$$

$$\Delta\rho_0(n) = \frac{\rho_0(n+1)p(n)}{r(n)}$$

$$\Delta\omega(n) = \rho_0(n+1)\Delta((r(n))|\Delta y(n)|^{\alpha-1}\Delta y(n))+$$

$$r(n)(|\Delta y(n)|^{\alpha-1}\Delta y(n)\frac{\rho_0(n+1)p(n)}{r(n)})$$

$$\begin{aligned}\Delta\omega(n) &= \rho_0(n+1)(\Delta(r(n))|\Delta y(n)|^{\alpha-1}\Delta(y(n)))\\ &\quad + p(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)\rho_0(n+1)\\ &= \rho_0(n+1)(\Delta(r(n))|\Delta y(n)|^{\alpha-1}\Delta(y(n)))\\ &\quad + p(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)\\ &= \rho_0(n+1)(-(q(n))f(y(n)))\\ \frac{\Delta\omega(n)}{f(y(n))} &= -q(n)\rho_0(n+1), \end{aligned} \tag{15}$$

$$\begin{aligned}\Delta\Big(\frac{\omega(n)}{f(y(n))}\Big) &= \frac{f(y(n))\Delta\omega(n)-\omega(n)\Delta f(y(n))}{f(y(n))f(y(n+1))}\\ \Delta\Big(\frac{\omega(n)}{f(y(n))}\Big) &= \frac{f(y(n))}{f(y(n+1))}\Big(-q(n)\rho_0(n+1)\Big)-\frac{\omega(n)\Delta f(y(n))}{f(y(n))f(y(n+1))}\\ \Delta\Big(\frac{\omega(n)}{f(y(n))}\Big) &= \frac{-q(n)\rho_0(n+1)f(y(n))}{f(y(n+1))}-\frac{\omega(n)\Delta f(y(n))}{f(y(n))f(y(n+1))}. \end{aligned} \tag{16}$$

Applying summation on both the sides

$$\frac{\omega(n)}{f(y(n))}-\frac{\omega(n_0)}{f(y(n_0))}=\sum_{j=n_0}^{n-1}\Big[\frac{-q(j)\rho_0(j+1)f(y(j))}{f(y(j+1))}-\frac{\omega(j)\Delta f(y(j))}{f(y(j))f(y(j+1))}\Big],$$

$$\frac{\omega(n)}{f(y(n))}=\frac{\omega(n_0)}{f(y(n_0))}-\Big[\sum_{j=n_0}^{n-1}\frac{-q(j)\rho_0(j+1)f(y(j))}{f(y(j+1))}+\frac{\omega(j)\Delta f(y(j))}{f(y(j))f(y(j+1))}\Big],$$

$$\begin{aligned}-\frac{\omega(n)}{f(y(n))} &= -\frac{\omega(n_0)}{f(y(n_0))}\\ &\quad+\Big[\sum_{j=n_0}^{n-1}\frac{-q(j)\rho_0(j+1)f(y(j))}{f(y(j+1))}+\frac{\omega(j)\Delta f(y(j))}{f(y(j))f(y(j+1))}\Big]. \end{aligned} \tag{17}$$

Using (8) that is $\sum\limits_{j=n_0}^{\infty} q(j)\rho(j) = \infty$. Choosing the positive quantity M so $M > 0$

$$-\frac{\omega(n_0)}{f(y(n_0))}+\sum_{j=n_0}^{n-1}\frac{q(j)\rho_0(j+1)f(y(j))}{f(y(j+1))}+\sum_{j=n_0}^{n_1}\frac{\omega(j)\Delta f(y(j))}{f(y(j))f(y(j+1))}, \tag{18}$$

(17) becomes

$$-\frac{\omega(n)}{f(y(n))} \geq M + \sum_{j=n_1}^{n-1} \frac{\omega(j)\Delta f(y(j))}{f(y(j))f(y(j+1))}. \tag{19}$$

Since $y(n)$ is positive, then (19) implies $-\omega(n) > 0$ or $\Delta y(n) < 0$. Let

$$g(n) = -\omega(n) = \rho_0(n)r(n)(-\Delta y(n))^{\alpha}, \tag{20}$$

then (19) becomes

$$\begin{aligned} g(n) &\geq Mf(y(n)) + \sum_{j=n_1}^{n-1} \frac{\omega(j)\Delta f(y(j))}{f(y(j))f(y(j+1))} f(y(n)) \\ &\geq Mf(y(n)) + \sum_{j=n_1}^{n-1} \frac{-g(j)\Delta f(y(j))}{f(y(j))f(y(j+1))} f(y(n)) \\ &\geq Mf(y(n)) + \sum_{j=n_1}^{n-1} \frac{g(j)(-\Delta f(y(j)))}{f(y(j))f(y(j+1))} f(y(n)). \end{aligned} \tag{21}$$

Let

$$\mathbb{K}(k, j, g) = \frac{f(y(n))(-\Delta f(y(j)))}{f(y(j))f(y(j+1))} g. \tag{22}$$

Then for fixed k and j $\mathbb{K}(k, j, g)$ is nondecreasing in g. Let $v(n)$ be the minimal solution of the equation

$$v(n) = Mf(y(n)) + \sum_{j=n_1}^{n-1} \frac{f(y(n))(-\Delta f(y(n)))}{f(y(j))f(y(j+1))} v(j). \tag{23}$$

By Lemma 2.1,

$$g(n) \geq v(n),$$

$$\frac{v(n)}{f(y(n))} = M + \sum_{j=n_1}^{n-1} \frac{(-\Delta f(y(j)))v(j)}{f(y(j))f(y(j+1))}.$$

Applying Δ on both the sides,

$$\Delta\Big[\frac{v(n)}{f(y(n))}\Big] = \Delta M + \Delta \sum_{j=n_1}^{n-1} \frac{(-\Delta f(y(j)))v(j)}{f(y(j))f(y(j+1))},$$

$$\Delta\Big[\frac{v(n)}{f(y(n))}\Big] = \frac{(-\Delta f(y(n)))v(n)}{f(y(n))f(y(n+1))}. \tag{24}$$

On the other hand,

$$\Delta\Big[\frac{v(n)}{f(y(n))}\Big] = \frac{f(y(n))\Delta v(n) - v(n)\Delta(f(y(n)))}{f(y(n+1))f(y(n))}$$

$$= \frac{\Delta v(n)}{f(y(n))} - \frac{v(n)\Delta f(y(n))}{f(y(n+1))f(y(n))}. \tag{25}$$

Combining (24) and (25),

$$\Delta\Big[\frac{v(n)}{f(y(n))}\Big] = \frac{v(n)(-\Delta f(y(n)))}{f(y(n))f(y(n+1))} = \frac{\Delta v(n)}{f(y(n))} - \frac{v(n)\Delta f(y(n))}{f(y(n+1))f(y(n))}$$

$$\frac{\Delta v(n)}{f(y(n+1))} = 0$$

$$\Delta v(n) = 0$$

$$v(n+1) = v(n)$$

$$v(n_1) = v(n).$$

By (23)

$$v(n_1) = Mf(u(n_1)) + \sum_{j=n_1}^{k} \frac{f(y(n))(-\Delta f(y(j)))}{f(y(j))f(y(j+1))} v(j)$$

$$v(n_1) = mf(u(n_1))$$

$$g(n) \geq v(n) \quad \textit{but} \qquad v(n) = v(n_1)$$

$$g(n) = \rho_0(n)r(n)(-\Delta y(n))^{\alpha} \geq mf(u(n_1))$$

$$(-\Delta y(n))^{\alpha} \geq \frac{mf(u(n_1))}{\rho_0(n)r(n)}$$

$$(-\Delta y(n)) \geq \Big(\frac{mf(u(n_1))}{\rho_0(n)r(n)}\Big)^{\frac{1}{\alpha}}$$

$$(-\Delta y(n)) \geq (mf(u(n_1)))^{\frac{1}{\alpha}} \Big[\frac{1}{\rho_0(n)r(n)}\Big]^{\frac{1}{\alpha}}.$$

Taking summation on both the sides,

$$-\sum_{j=n_1}^{n-1} \Big(u(j+1) - u(j)\Big) \geq (mf(u(n_1)))^{\frac{1}{\alpha}} \sum_{j=n_1}^{n-1} \Big[\frac{1}{\rho_0(n)r(n)}\Big]^{\frac{1}{\alpha}} \tag{26}$$

as $n \to \infty$ (26), and using (14) implies that $y(n) \leq -\infty$. This is a contradiction. Hence $y(n)$ is oscillatory. $\square$

In the following, we always suppose that $H(t) \in C^2(R,R)$ and it satisfies the following conditions:

(H_1) $H(t) > 0$ for $t \geq t_0$, $H(t)$ is bounded function;

(H_2) $\Delta H(t) = h(t), h(t)$ is bounded function.

Theorem 2.6. *Assume that* $\Delta f(u) \geq 0$, $\sum\limits_{k=n_0}^{\infty} \frac{1}{r^{1/\alpha}(k)} = \infty$ *holds*

$$p(n) \leq 0; \qquad q(n) > 0, \tag{27}$$

or

$$p(n) \leq 0; \qquad q(n) \leq 0 \quad \lim_{n\to\infty} \frac{p(n)}{q(n)} = M > 0. \tag{28}$$

Suppose that there exists a function $H(t)$ *that satisfies*

$$\sum_{j=n_0}^{\infty} H(j)\varphi(j) = \infty, \tag{29}$$

$$\lim_{n\to\infty} \sup\ v(n)r(n) < \infty, \tag{30}$$

where

$$\varphi(n) = v(n+1)(q(n) - \Delta r(n)h(n)) - h(n)p(n), \tag{31}$$

$$v(n) = \sum_{j=n_0}^{n-1} \Big(\frac{p(j)}{r(j)} - \frac{h(j)}{H(j)}\Big). \tag{32}$$

Then every solution of (1) is oscillatory.

Proof. For the sake of contrary, (1) has a non oscillatory solution $y(n)$. Without loss of generality, we assume that $y(n) > 0$ for all $n \geq n_0$.

Let

$$v(n) = \sum_{j=n_0}^{n-1} \Big[\frac{p(j)}{r(j)} - \frac{h(j)}{H(j)}\Big], \tag{33}$$

$$\Delta v(n) = \sum_{j=n_0}^{n} \Big[\frac{p(j)}{r(j)} - \frac{h(j)}{H(j)}\Big] - \sum_{j=n_0}^{n-1} \Big[\frac{p(j)}{r(j)} - \frac{h(j)}{H(j)}\Big], \tag{34}$$

$$\Delta v(n) = \frac{p(n)}{r(n)} - \frac{h(n)}{h(n)}. \tag{35}$$

Define

$$\ell(n) = v(n)r(n)\Big[\frac{|\Delta y(n)|^{\alpha-1}\Delta y(n)}{f(y(n))} + h(n)\Big]$$
$$= v(n)\Big(\frac{r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)}{f(y(n))} + r(n)h(n)\Big),$$

$$\Delta\ell(n) = \Delta v(n)\Big[\frac{r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)}{f(y(n))} + r(n)h(n)\Big] + v(n+1)$$
$$\Big(\frac{f(y(n))\Delta(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n))}{f(y(n))f(y(n+1))}$$
$$-\frac{(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)\Delta f(y(n)))}{f(y(n))f(y(n+1))} + \Delta r(n)h(n)\Big)$$

$$\Delta\ell(n) = \Big[\frac{p(n)}{r(n)} - \frac{h(n)}{H(n)}\Big]\frac{\ell(n)}{v(n)}$$
$$+ v(n+1)\Big[\frac{\Delta(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n))}{f(y(n+1))}$$
$$-\frac{r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)\Delta f(y(n))}{f(y(n))f(y(n+1))} + \Delta r(n)h(n)\Big]$$
$$\leq -\frac{h(n)}{H(n)}\Big[\frac{\ell(n)}{v(n)}\Big] + p(n)h(n) +$$
$$v(n+1)\Big(\frac{-p(n)|\Delta y(n)|^{\alpha-1}\Delta y(n) - q(n)f(y(n))}{f(y(n+1))}$$
$$-\frac{r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)\Delta f(y(n))}{f(y(n))f(y(n+1))} + \Big(\Delta r(n)h(n)\Big)\Big)$$
$$\leq -\frac{h(n)}{H(n)}\Big[\frac{\ell(n)}{v(n)}\Big] + p(n)h(n) +$$
$$v(n+1)\left\{\frac{-p(n)|\Delta y(n)|^{\alpha-1}\Delta y(n) - q(n)f(y(n))}{f(y(n+1))}\right.$$
$$\left.-\frac{r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)\Delta f(y(n))}{f(y(n))f(y(n+1))} - \Delta r(n)h(n)\right\}$$

$$\leq -\frac{h(n)}{h(n)}\Big[\frac{\ell(n)}{v(n)}\Big] + p(n)h(n) +$$
$$v(n+1)\Big[\frac{-p(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)}{f(y(n+1))}$$
$$-\frac{q(n)f(y(n))}{f(y(n+1))} - \frac{r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)\Delta f(y(n))}{f(y(n))f(y(n+1))} + \Delta r(n)h(n)\Big]$$
$$\leq -\frac{h(n)}{H(n)}\Big(\frac{\ell(n)}{v(n)}\Big) + p(n)h(n) - \frac{v(n+1)q(n)f(y(n))}{f(y(n+1))} + \Delta r(n)h(n)v(n+1)$$
$$\leq -\frac{h(n)}{H(n)}\Big(\frac{\ell(n)}{v(n)}\Big) + p(n)h(n) - q(n)v(n+1) + \Delta r(n)h(n)v(n+1)$$

$$\leq -\frac{h(n)}{H(n)}\Big(\frac{\ell(n)}{v(n)}\Big) - \varphi(n)$$
$$H(n)\Delta\ell(n) \leq -h(n)\ell(n) - \varphi(n)H(n)$$
$$H(n)\Delta\ell(n) \leq -h(n)\ell(n) - H(n)\varphi(n)$$
$$H(n)\varphi(n) \leq -h(n)\ell(n) - H(n)\Delta\ell(n). \tag{36}$$

Case (i): $(\ell(n))$ oscillatory. Then there exists a sequence $\{k_n\}$, $n = 1, 2, 3, ...$, ∞ as $n \to \infty$ and s.t. $\ell(k_n) = 0, n = 1, 2, 3\cdots$. Finding summation of (36) on both sides, we get

$$\sum_{j=n_0}^{k_n-1} H(j)\varphi(j) \leq -\sum_{j=n_0}^{k_n-1}\Big[h(j)\ell(j) + H(j)\Delta\ell(j)\Big].$$

From the product formula, we find

$$\sum_{=1}^{n-1}(y_n\Delta x(n)) = \Big[x_k, y_k\Big] - \sum_{r=1}^{n-1} x_{r+1}\Delta y_r + c$$
$$\leq -\sum_{j=n_0}^{k_n-1}\Big[h(j)\ell(j)\Big] - \sum_{j=n_0}^{k_n-1} H(j)\Delta\ell(j)$$
$$\leq -\sum_{j=n_0}^{k_n-1} h(j)\ell(j) - \left\{H(k_n)\ell(k_n) - \sum_{j=n_0}^{k_n-1} H(j+1)\Delta\ell(j)\right\}$$
$$\leq -\sum_{j=n_0}^{k_n-1}\Big[h(j)\ell(j) + H(j+1)\Delta\ell(j)\Big] - H(k_n)\ell(k_n)$$
$$\leq -\Big[H(k_n)\ell(k_n)\Big]$$

$$\times \lim_{k_n \to \infty} \sum_{j=n_0}^{k_n - 1} H(j)\varphi(j) \le -H(k_n)\ell(k_n) \le 0.$$

This is a contradiction to (29).

Case (ii) $\ell(n)$ is eventually positive. Taking summation on both sides of (36), we get

$$\sum_{j=n_0}^{k_n - 1} H(j)\varphi(j) \le -H(k_n)\ell(k_n),$$

which is a contradiction.

Case (iii) $\ell(n)$ is eventually negative, $\underline{\lim}\sup\ \ell(n) > -\infty$. Then there exists a sequence $\overline{k_n}$, $n = 1, 2, 3...$ satisfies $\overline{k_n} \to \infty$ as $n \to \infty$ and such that $\underline{\lim}_{\overline{k_n} \to \infty} \ell(\overline{k_n} = \limsup_{k \to \infty} y(n) = M_1 > -\infty$. Because $H(n)$ is bounded function then there exists a $M_2 > 0$; s.t. $H(\overline{k_n}) \le M_2; n = 1, 2, 3, \ldots$. According to (36), we obtain

$$\begin{aligned} \sum_{j=n_0}^{\overline{k_n} - 1} H(j)\varphi(j) &\le -\sum_{j=n_0}^{\overline{k_n} - 1} \Big(h(j)\ell(j) + H(j+1)\Delta\ell(j)\Big) - \Big(H(\overline{k}_n)\ell(\overline{k_n})\Big) \\ &\le -\Big(H(\overline{k_n})\ell(\overline{k_n})\Big). \end{aligned}$$

Taking limit $(\overline{k_n}) \to \infty$

$$\infty \le M_1 M_2.$$

This is obviously a contradiction. □

Theorem 2.7. *Assume that (30) holds,* $\Delta f(u) \ge 0$, $\sum\limits_{k=n_0}^{\infty} \frac{1}{r^{1/\alpha}(k)} = \infty$ *and*

$$p(n) \le 0, \qquad q(n) > 0, \tag{37}$$

or

$$p(n) \le 0, \qquad q(n) \le 0 \qquad \lim_{n \to \infty} \frac{p(n)}{q(n)} = M > 0. \tag{38}$$

Suppose further that there exists a function $H(n)$ *that satisfies* (H_1), (H_2) *and s.t.*

$$\sum_{j=n_0}^{\infty} H(n)(\overline{\varphi}(n)) = \infty, \tag{39}$$

where

$$\overline{\varphi}(n) = v(n+1)(q(n) + \Delta r(n)h(n)) + h(n)p(n) \tag{40}$$

and $v(n)$ is defined in (32). Then, every solution of (1) is oscillatory.

Proof. For the sack of contrary (1) has a nonoscillatory solution. Without loss of generality we may assume that (1) has an eventually positive solution (if it has an eventually negative solution, the proof is similar), then there exists a $n_1 > n_0$ s.t. $y(n) > 0$ for all $n \geq n_1$. Define

$$\ell(n) = v(n)r(n)\Big[\frac{|\Delta|^{\alpha-1}\Delta y(n)}{f(y(n))} - h(n)\Big]. \tag{41}$$

The rest of the proof is similar to that of Theorem 2.6. The proof is completed. □

Theorem 2.8. *Assume that $\Delta f(u) \geq 0$, $\sum\limits_{k=n_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(k)} = \infty$ hold,*

$$p(n) < 0; \qquad q(n) > 0, \tag{42}$$

$$p(n) \leq 0; \qquad q(n) \leq 0 \qquad \lim_{n\to\infty} \frac{p(n)}{q(n)} = M > 0. \tag{43}$$

Suppose further that there exists a function $H(n)$ that satisfies (H_1), (H_2) and s.t.

$$\sum_{j=n_0}^{\infty} H(n)\varphi(n) = \infty, \tag{44}$$

$$\varphi(n) = v(n+1)\Big[-\Delta r(n)h(n)\Big] - p(n)h(n), \tag{45}$$

where $v(n)$ is defined in (32). Then every solution of (1) is oscillatory.

Proof. For the sake of contrary, (1) has a non oscillatory solution $y(n)$. Without loss of generality, we may assume that $y(n) > 0$ for all $k > n_0$. Define

$$\begin{aligned}\ell(n) &= v(n)r(n)\Big[\frac{|\Delta y(n)|^{\alpha-1}\Delta y(n)}{y(n)} + h(n)\Big] \\ &= v(n)\Big(\frac{r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)}{y(n)} + r(n)h(n)\Big).\end{aligned} \tag{46}$$

Nothing that $y(n)f(y(n)) \geq 0 \quad$ for $y(n) \neq 0$

$$\begin{aligned}
\Delta\ell(n) &= \Delta v(n)\Big[\frac{r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)}{y(n)} + r(n)h(n)\Big] \\
&\quad + v(n+1)\Big(\frac{y(n)(\Delta(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)))}{(y(n)y(n+1))} \\
&\quad - \frac{(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)\Delta y(n))}{(y(n)y(n+1))} + \Delta r(n)h(n)\Big) \\
&= \Big[\frac{p(n)}{r(n)} - \frac{h(n)}{H(n)}\Big]\frac{\ell(n)}{v(n)} + v(n+1) \\
&\quad \Big[\frac{\Delta(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n))}{y(n+1)} - \frac{(r(n)|\Delta y(n)|^{\alpha-1})\Delta y(n)\Delta y(n)}{y(n)y(n+1)} \\
&\quad + \Delta r(n)h(n)\Big] \\
&= -\frac{h(n)}{H(n)}.\frac{\ell(n)}{v(n)} + \frac{p(n)}{r(n)}.\frac{\ell(n)}{v(n)} + v(n+1) \\
&\quad \Big[\frac{y(n)\Delta(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)) - (r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n))}{y(n+1)y(n)} \\
&\quad + \Delta r(n)h(n)\Big]
\end{aligned}$$

$$\begin{aligned}
\Delta\ell(n) &\leq -\frac{h(n)}{H(n)}.\frac{\ell(n)}{v(n)} + p(n)h(n) + v(n+1)\Big[\frac{-p(n)|\Delta y(n)|^{\alpha-1}\Delta y(n)}{y(n+1)} \\
&\quad - \frac{q(n)f(y(n))}{y(n+1)} - \frac{(r(n)|\Delta y(n)|^{\alpha-1}\Delta y(n))}{y(n)y(n+1)} + \Delta r(n)h(n)\Big] \\
&\leq -\frac{h(n)}{h(n)}.\frac{\ell(n)}{v(n)} + p(n)h(n) + \Delta r(n)h(n)v(n+1) \\
\Delta\ell(n) &\leq -\frac{h(n)}{H(n)}\ell(n) + p(n)h(n) + \Delta r(n)h(n)v(n+1)
\end{aligned}$$

$$\ell(n) \leq -\frac{h(n)}{H(n)}\ell(n) - \varphi(n) \tag{47}$$

$$\Delta\ell(n)H(n) \leq -h(n)\ell(n) - \varphi(n)H(n)$$

$$H(n)\varphi(n) \leq -h(n)\ell(n) - \Delta\ell(n)H(n). \tag{48}$$

The rest of the proof is similar to that of Theorem 2.6, hence the proof is completed. □

Example 2.9. Consider the following difference equation

$$\Delta\Big[t^4\Delta x(t)\Big] - 2\Delta x(t) + \Big(t + \frac{3}{4t}\Big)x(t) = 0.$$

It is obvious that $\alpha = 1; q(t) = t + \frac{3}{4t}; p(t) = -2$ and $r(t) = t^4$. It is difficult to distinguish whether every solution of the above equation is oscillatory. But by Theorem 2.6, we find

$$(i) \quad \sum_{j=t_0}^{\infty} q(j)\rho(j) = \sum_{j=t_0}^{\infty} \Big(j + \frac{3}{4j}\Big)\Big(\frac{1}{j^2}\Big),$$

$$(ii) \quad -2\Big(\frac{1}{t^2}\Big) \geq \frac{t^4(-2t+1)}{t^2(1+t)^2}$$

$$-2(\frac{1}{t^2}) \geq \frac{-(2t+1)}{1+2/t+1/t^2}.$$

Hence by the theorem, we can easily show that the given equation is oscillatory.

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