International Journal of Applied Mathematics

Volume 25 No. 2 2012, 183-190

THE COHOMOLOGICAL INVARIANT $E'(G,W)$ AND SOME PROPERTIES

Maria Gorete Carreira Andrade[1 §], Ermínia de Lourdes Campello Fanti[2]

[1,2]Department of Mathematics - IBILCE
UNESP - São Paulo State University
R. Cristovão Colombo, 2265
15054 - 000 - São José do Rio Preto-SP, BRAZIL
[1]e-mail: gorete@ibilce.unesp.br
[2]e-mail: fanti@ibilce.unesp.br

Abstract: Let G be a group, W a nonempty G-set and M a $\mathbb{Z}_2G$-module. Consider the restriction map $res^G_W : H^1(G,M) \to \prod_{w_i \in E} H^1(G_{w_i},M)$, $[f] \to (res^G_{G_{w_i}}[f])_{i\in I}$, where $E = \{w_i,\ i \in I\}$ is a set of orbit representatives in W and $G_{w_i} = \{g \in G \mid gw_i = w_i\}$ is the G-stabilizer subgroup (or isotropy subgroup) of w_i, for each $w_i \in E$. In this work we analyze some results presented in Andrade et al [5] about splittings and duality of groups, using the point of view of Dicks and Dunwoody [10] and the invariant $E'(G,W) := 1 + \dim \ker res^G_W$, defined when G_{w_i} is a subgroup of infinite index in G for all w_i in E, and $M = \mathbb{Z}_2$ (where $\dim = \dim_{\mathbb{Z}_2}$). We observe that the theory of splittings of groups (amalgamated free product and HNN-groups) is inserted in the combinatory theory of groups which has many applications in graph theory (see, for example, Serre [12] and Dicks and Dunwoody [10]).

AMS Subject Classification: 20E06, 05C25, 20J06
Key Words: cohomology of groups, duality, splittings of groups

Received: January 22, 2012

§Correspondence author

1. Introduction

Andrade and Fanti [1] defined a generalized invariant $E(G, \mathcal{S}, M)$, where G is a group, $\mathcal{S}$ is a family of subgroups of infinite index in G and M is a $\mathbb{Z}_2G$-module, and studied the particular case where $\mathcal{S} = \{S\}$, a family with only one subgroup, and $M = \mathbb{Z}_2(G/S)$. Some properties were considered in the general case in Andrade et al [4]. Also, considering the particular modules $\mathbb{Z}_2$, $\mathbb{Z}_2G$ and $\mathbb{Z}_2G \otimes_{\mathbb{Z}_2S} \overline{\mathbb{Z}_2S} = \mathcal{F}_SG$ (Andrade et al [5], Andrade et al [4] and Andrade and Fanti [2]), some results were obtained. In the definition of the invariant $E(G, \mathcal{S}, M)$ we used the relative cohomology theory for the pair $(G, \mathcal{S})$ (given by Bieri and Eckmann [8]). Dicks and Dunwoody [10] worked with relative cohomology from another point of view. They worked with a G-set W and the pair (G, W). These two approaches of the relative cohomology theory are very interesting in the areas of Algebra and Algebraic Topology, because many results on splittings of groups and duality obtained using the theory of Bieri-Eckmann can be achieved using the elegant theory of Dicks-Dunwoody. For this it is necessary to see the relationship between the two definition of relative cohomology. This is done in detail in Andrade et al [6], and will be used here. In this work we define the invariant $E(G, W, M)$ using the theory of cohomology of Dicks and Dunwoody and we analyze some results presented in Andrade et al [5] about splittings and duality of groups using the invariant $E(G, W, \mathbb{Z}_2)$ for pairs (G, W), with some estrictions. We observe that the theory of splittings of groups has many applications in several areas of pure and applied mathematics. Among them, we highlight the graph theory (Serre [10] and Dicks and Dunwoody [12]).

2. The Cohomological Invariant $E'(G, W)$

In this section we give the definition and some properties of the invariant $E'(G, W)$. Before that, we recall the definition of relative cohomology according to Dicks-Dunwoody [10].

Definition 1. Consider W a G-set and $\mathbb{Z}_2W$ the free $\mathbb{Z}_2$-module generated by W. Let $\varepsilon : \mathbb{Z}_2W \to \mathbb{Z}_2$ be the augmentation map, $\Delta = \ker \varepsilon$, M a $\mathbb{Z}_2G$-module and $P \to \Delta$ a projective $\mathbb{Z}_2G$ resolution of Δ. The groups of relative homology and cohomology of the pair (G, W), with coefficients in M, for all $k \in \mathbb{Z}$, are respectively defined by $H_k(G, W; M) = H_k(P \otimes_{\mathbb{Z}_2G} M)$, and $H^k(G, W; M) = H^k(Hom_{\mathbb{Z}_2G}(P, M))$.

The next result will be useful in the definition of the invariant $E'(G,W)$ and in the proof of its properties.

Proposition 1. *Let (G,W) be a pair where G is a group and W is a G-set. Consider E a set of orbit representatives in W and G_w the G-stabilizer of w, for each $w \in E$, and M a $\mathbb{Z}_2G$-module. Denote $\prod\limits_{w\in E} H^k(G_w;M)$ by $H^k(\mathcal{W};M)$ for all $k \in \mathbb{Z}$. Then, we have the long exact sequence*

$$0 \to H^0(G;M) \xrightarrow{\chi} H^0(\mathcal{W};M) \xrightarrow{l} H^1(G,W;M) \xrightarrow{J}$$
$$\xrightarrow{J} H^1(G;M) \xrightarrow{res^G_W} H^1(\mathcal{W};M) \to \cdots$$

which is natural in the module M and in the pair (G,W).

Proof: For the group G, let M be a $\mathbb{Z}_2G$-module and W a G-set. Consider $E = \{w_i,\ i \in I\}$ the set of orbit representatives in W and let $\mathcal{S} = \{G_{w_i} \mid i \in I\}$ be the family of isotropy subgroups. We know that the relative (co)homology groups of the pair (G,W) with coefficients in M and the relative (co)homology groups of the pair $(G,\mathcal{S})$ with coefficients in M are isomorphic, i. e., for all $k \in \mathbb{Z}$, we have $H_k(G,W;M) \simeq H_k(G,\mathcal{S};M)$ and $H^k(G,W;M) \simeq H^k(G,\mathcal{S};M)$ (details in Andrade et al [6]). So the result follows from the long exact sequence for the pair $(G,\mathcal{S})$ (Bieri and Eckmann [8]). □

Definition 2. Let (G,W) be a pair where G is a group and W is a G-set such that $[G:G_w]=\infty$ for all $w \in E$, where E is a set of orbit representatives in W, and M is a $\mathbb{Z}_2G$ -module. We define $E(G,W,M) = 1+\dim\ker res^G_W$. When M is the trivial $\mathbb{Z}_2G-$module $\mathbb{Z}_2$ we will denote, for simplicity, $E(G,W,\mathbb{Z}_2)$ by $E'(G,W)$.

Proposition 2. *Let (G,W) be a pair as in the former definition. If the $\mathbb{Z}_2$ - vector spaces $H^0(G;M)$, $H^0(\mathcal{W};M) := \prod\limits_{i\in I} H^0(G_{w_i};M)$ and $H^1(G,W;M)$ have finite dimensions, then*

$$E(G,W,M) = 1 + \dim H^0(G;M) - \dim H^0(\mathcal{W};M) + \dim H^1(G,W;M).$$

Proof: The proof follows immediately from the long exact sequence given in the previous proposition. □

Proposition 3. *Let G be a group and W a G-set with W infinite. If the G-action is transitive then*

$$E'(G,W) = 1 + \dim H^1(G, G_w; \mathbb{Z}_2),$$

where w is any element of W.

Proof: Since the $G-$action is transitive, $G(w) = W$ for all $w \in W$, and so any $w \in W$ is a representative for the single G-orbit W. Let $w \in W$ and $E = \{w\}$. Consider G_w the G-stabilizer of w. Using the bijection between $W = G(w)$ and G/G_w, and the fact that W is infinite, we obtain $[G : G_w] = \infty$. Now, $H^0(G;\mathbb{Z}_2) \simeq \mathbb{Z}_2^G \simeq \mathbb{Z}_2$, and $H^0(G_w;\mathbb{Z}_2) \simeq \mathbb{Z}_2^{G_w} \simeq \mathbb{Z}_2$. Then, the application χ in the long exact sequence

$$0 \to H^0(G;\mathbb{Z}_2) \xrightarrow{\chi} H^0(G_w;\mathbb{Z}_2) \xrightarrow{l} H^1(G,W;\mathbb{Z}_2) \xrightarrow{J}$$
$$\xrightarrow{J} H^1(G;\mathbb{Z}_2) \xrightarrow{res_W^G} H^1(G_w;\mathbb{Z}_2) \to \cdots$$

is an isomorphism, and so J is injective. Thus $H^1(G,W;\mathbb{Z}_2) \simeq \ker res_S^G$ and therefore, $E'(G,S) = 1 + \dim H^1(G,W;\mathbb{Z}_2)$. $\square$

Proposition 4. *Let G be an infinite group and W a G-set. If the G-action is free and $E = \{w_i,\ i \in I\}$ is a set of G-orbit representatives in W, then $E'(G,W) = 1 + \dim H^1(G;\mathbb{Z}_2)$.*

Proof: Since the action is free, we have $G_w = \{1\}$, for all $w \in E$, and therefore $H^1(G_w;\mathbb{Z}_2) = H^1(\{1\};\mathbb{Z}_2) = 0$, for all w. Hence, $ker\ res_W^G = H^1(G;\mathbb{Z}_2)$ and so $E'(G,W) = 1 + \dim H^1(G;\mathbb{Z}_2)$. Note that $[G : G_w] = |G| = \infty$, for all w in W. $\square$

Example 1. Consider $G =< a > \cong \mathbb{Z}$ and $W = \mathbb{R}$ with the G-action, $a^k.w = k + w$. A set of G-orbit representatives in W is $E = [0,1[$, since for each $r \in \mathbb{R}$, there are $w \in [0,1[$ and $n \in \mathbb{Z}$ such that $r = w+n$, and so $G(r) = G(w) = \{w+n, n \in \mathbb{Z}\}$. This action is free, so $G_w = \{1\}$, for all $w \in E$ and $\{G_w\}_{w\in E} = \{\{1\}_{w\in E}\}$. By the last proposition, $E(G,\mathbb{R},\mathbb{Z}_2) = 1 + \dim H^1(G;\mathbb{Z}_2) = 2$, because $H^1(G;\mathbb{Z}_2) = {\mathbb{Z}_2}^G = \mathbb{Z}_2$.

3. Some Results in Duality and Splittings

First, we will explore some relations between $E'(G,W)$ and duality pairs.

Definition 3. Let (G,W) be a pair where G is a group and W is a G-set. (G,W) is a Poincaré duality pair of dimension n, or simply a PD^n-pair *(*over $\mathbb{Z}_2$*)*, if there exist natural isomorphisms $H^k(G,W;M) \simeq H_{n-k}(G,M)$, and $H_k(G,W;M) \simeq H^{n-k}(G,M)$, for all $\mathbb{Z}_2G$-modules M and all $k \in \mathbb{Z}$.

Remark 1. (Kropholler and Roller, see [11] p. 273) Let (G,W) be a PD^n-pair.
i) If $W = \emptyset$ then G is a duality group.
ii) W falls into finitely many G-orbits.
iii) For each $w \in W$, the stabilizer subgroup G_w is a PD^{n-1}-group.

Theorem 4. *Let (G,W) be a PD^n-pair, $E = \{w_i,\ i = 1,...,r\}$ a set of orbit representatives in W, $\mathcal{S} = \{G_{w_i} \mid i = 1,...,r\}$ the family of $G-$stabilizer subgroups. Suppose that $[G : G_{w_i}] = \infty$ for all $w_i \in E$ and G is finitely presented. Then $E'(G,W)$ is finite and*

$$E'(G,W) = 2 - r + \dim H_{n-1}(G;\mathbb{Z}_2).$$

Proof : If (G,W) is a PD^n-pair then $(G,\mathcal{S})$ is a PD^n-pair (Andrade et al [6]), where $\mathcal{S} = \{G_{w_i},\ i = 1,\ldots,r\}$, and we have that G is a D^{n-1}-group and consequently, by Bieri [7], Theorem 9.2, G is of type FP . Since G is finitely presented, there is a finitely dominated $K(G,1)$-complex (Brown [9], VIII.7.1). Thus, $H_*(G;\mathbb{Z}_2) = H_*(K(G,1);\mathbb{Z}_2)$ are finitely generated. Now we have $H^0(G;\mathbb{Z}_2) \simeq \mathbb{Z}_2$, $\prod_{i=1}^{r} H^0(G_{w_i};\mathbb{Z}_2) \simeq \bigoplus_{i=1}^{r} H^0(G_{w_i};\mathbb{Z}_2) \simeq \bigoplus_{i=1}^{r} \mathbb{Z}_2$ and by duality $H^1(G,W;\mathbb{Z}_2) \simeq H_{n-1}(G;\mathbb{Z}_2)$. Since $\dim H_{n-1}(G;\mathbb{Z}_2) < \infty$, we have, by Proposition 2, for $M = \mathbb{Z}_2$, $E'(G,W) = E(G,W,\mathbb{Z}_2) = 1 + (1 - r + \dim H_{n-1}(G;\mathbb{Z}_2)) < \infty$. □

Corollary 5. *If (G,W) is a PD^n-pair, $E = \{w_i,\ i = 1,...,r\}$ is a set of orbit representatives in W, $\{G_{w_i} \mid i = 1,...,r\}$ is the family of G-stabilizer subgroups with $[G : G_{w_i}] = \infty$ for all $w_i \in E$ and G is finitely presented, then $|E| = r \leq 1 + \dim H_{n-1}(G;\mathbb{Z}_2)$.*

Corollary 6. *If $G = \mathbb{Z} * \mathbb{Z}$ and there exists a G-set W such that (G,W)*

is a PD^2-pair, with $[G : G_{w_i}] = \infty$ for all $w_i \in E$, where $E = \{w_i, i = 1, ..., r\}$ is a set of G-orbit representatives in W and $\{G_{w_i} \mid i = 1, ..., r\}$ is the family of stabilizer subgroups, then $|E| \leq 3$.

Proof: This follows from the fact that $G = \mathbb{Z} * \mathbb{Z}$ is a D^1-group (Brown [9]) and $H_1(G; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. $\square$

Now, we recall when a group G splits over a subgroup T.

Definition 7. We say that a group G splits over a subgroup T if G is either a non trivial product with amalgamation over T, i.e., $G = G_1 *_T G_2$ with $G_1 \neq T \neq G_2$, or an HNN-group with base group T, i.e., $G = G_1 *_{T,\sigma}$.

The next result shows the values which $E'(G, W)$ can assume when G splits over a subgroup T, under certain conditions.

Theorem 8. *Let (G, W) be a pair where G is a group which splits over a subgroup T and W is a nonempty $G-$set. Let E be a set of orbit representatives in W.*
(a) *If $G = G_1 *_T G_2$ and $E = \{w_1, w_2\}$ with $\{G_{w_1}, G_{w_2}\} = \{G_1, G_2\}$, then $E'(G, W) = 1$.*
(b) *If $G = G_1 *_{T,\sigma}$ and $E = \{w\}$ (the action is transitive) with $G_w = G_1$, then $E'(G, W) = 2$.*

Proof: If G splits over a subgroup T then $[G : S] = \infty$ for all subgroup S of G (Andrade and Fanti [3]). Thus the invariant E' can be defined.

(a) If $G = G_1 *_T G_2$, then one has the short exact sequence of $\mathbb{Z}_2 G$-modules (Bieri [7], Proposition 2.8, p. 27)

$$0 \to \mathbb{Z}_2(G/T) \xrightarrow{\alpha} \mathbb{Z}_2(G/G_1) \oplus \mathbb{Z}_2(G/G_2) \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0,$$

where α is given by $\alpha(xT) = (xG_1, -xG_2) = (xG_1, xG_2)$, and ε is the augmentation, $\varepsilon(xG_1, 0) = \varepsilon(0, xG_2) = 1,\ x \in G$. Then $\Delta = ker\varepsilon = Im\alpha \simeq \mathbb{Z}_2(G/T)$, and so, using that $H^1(G, W; \mathbb{Z}_2) \simeq H^1(G, \{G_1, G_2\}; \mathbb{Z}_2)$ and Shapiro's Lemma, we have $H^1(G, W; \mathbb{Z}_2) \simeq H^1(G, \{G_1, G_2\}; \mathbb{Z}_2) := H^0(G; Hom_{\mathbb{Z}_2}(\Delta, \mathbb{Z}_2)) \simeq H^0(T; \mathbb{Z}_2) = \mathbb{Z}_2$. Since $H^0(G; \mathbb{Z}_2)$ and $H^0(G_{w_i}; \mathbb{Z}_2),\ i = 1,\ 2$, are isomorphic to $\mathbb{Z}_2$, it follows from Proposition 2 that $E'(G, \{G_1, G_2\}) = 1 + 1 - 2 + 1 = 1$.

(b) The proof is similar to (a), using that if $G = G_1 *_{T,\sigma}$, with $G_1 *_{T,\sigma} =< X_1,\ p;\ R_1,\ p^{-1}tp = \sigma(t),\ t \in T >$, then we have the short exact sequence of

$\mathbb{Z}_2G$-modules (Bieri [7], Proposition 2.11, p. 32)

$$0 \to \mathbb{Z}_2(G/T) \xrightarrow{\alpha} \mathbb{Z}_2(G/G_1) \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0$$

where ε is induced by the augmentation and α is given by $\alpha(xT) = xG_1 - xpG_1$, p being the stable letter of G. □

References

[1] M.G.C. Andrade, E.L.C. Fanti, A relative cohomological invariant for pairs of groups, *Manuscripta Math.*, **83** (1994), 1-18.

[2] M.G.C. Andrade, E.L.C. Fanti, A note about splittings of groups and commensurability under a cohomological point of view, *Algebra and Discrete Mathematics*, **9** (2010), 1-10.

[3] M.G.C. Andrade, E.L.C. Fanti, A remark about amalgamation of groups and index of certain subgroups, *International Journal of Applied Mathematics*, **1** (2010), 55-62.

[4] M.G.C. Andrade, J.A. Daccach, E.L.C. Fanti, On certain relative cohomologic invariant, *International Journal of Pure and Applied Mathematics*, **21**, 3 (2005), 335-351.

[5] M.G.C. Andrade, E.L.C. Fanti, F.M.G. Papani, A relative invariant, duality and splitting of groups, *Revista de Matem. e Est.*, **21** (2003), 131-141.

[6] M.G.C. Andrade, E.L.C. Fanti, L.L. Fêmina, Some considerations about relative cohomology of groups and Poincaré duality pairs, *Submitted.*

[7] R. Bieri, *Homological Dimension of Discrete Groups*, Queen Mary College Notes, London (1976).

[8] R. Bieri, B. Eckmann, Relative homology and Poincaré duality for group pairs, *Journal of Pure and Applied Algebra*, **13** (1978), 277-319.

[9] K.S. Brown, *Cohomology of Groups*, Springer Verlag, G.T.M. 87, New York (1982).

[10] W. Dicks, M.J. Dunwoody, *Groups Acting on Graphs*, Cambridge University Press, Cambridge (1989).

[11] P.H. Kropholler, M.A. Roller, Splittings of Poincaré duality groups III, *J. London Math Soc. (2)*, **39** (1989), 271-284.

[12] J.-P. Serre, *Trees*, Springer-Verlag, New York (1980).