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THE ANALYTICAL INVARIANTS OF SPATIALLY HOMOGENEOUS ISOTROPIC BIANCHI TYPE I COSMOLOGICAL MODEL

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Abstract: It is seen that the spatially homogeneous isotropic Bianchi type I cosmological model is controlled by a single mathematical quantity ρ^3 or ρ^4. The expansion in the model decreases, when the value of ρ^3 or ρ^4 decreases and it becomes zero when ρ^3 or $\rho^4 \to 0$. The volume V attains the maximum value and it becomes infinite when ρ^3 or $\rho^4 \to 0$. There is no shear in the model. The pressure p and density ρ in the model are increasing with the values of ρ^3 and ρ^4. We get a vacuum model, when ρ^3 or $\rho^4 \to 0$.

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1. Introduction

The theory of general relativity has been developed with the advance of the investigation into spherically symmetric space-times. Many important space-times models dealing with relativistic theories are also spherically symmetric.

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Hence investigation into spherically symmetric space-times has been one of the important problems in relativistic theories. The general relativity is a theory of invariant relations under group of coordinate transformations. In this point of view the spherically symmetric space-times must be formulated invariantly. The theorem concerning geometrical and physical properties of spherically symmetric space-times becomes significant when they are stated in invariant forms.

Takeno [1] developed the theory of spherically symmetric space-times on the basis of invariant quantities containing two orthogonal unit vectors α_i and β_i and studied its properties. Thereafter, Borkar [2] (Ph.D. Thesis) developed the theory of **duplexes*** of orthogonal unit vectors α_i and β_i and studied the spherically symmetric space-times on the basis of it and investigated many results and discussed geometrical and physical properties of spherically symmetric space-times. Borkar and Hajare [3] studied orthogonal spherically symmetric space-times models by defining **triples**** of orthogonal unit vectors α_i and β_i and they shown that there are 48-numbers of triples of orthogonal unit vectors α_i β_i, out of which 24-triples reduced to duplexes, given in Borkar [2] and the remaining 24-number of triples (not reducing to duplexes) were studied with the geometrical and physical properties in it. Thereafter the spherically symmetric space-times, using the duplexes were examined by Borkar and Hajare [4]. The most general SSST formulated by Takeno [1], were studied and discussed for its geometrical and physical properties by Borkar and Hajare [5]. We apply this theory of triples of orthogonal unit vectors α_i β_i in the study of cosmological models of the universe.

Spatially homogeneous isotropic Bianchi type-I cosmological model in Einstein theory of gravitation, was deduced by Shahu et al. [6]. We take up this model in our discussion by applying the technique of triples of orthogonal unit vectors. Therefore, in this note, an attempt is made to study the spatially homogeneous isotropic Bianchi type-I cosmological model [6], on the basis of triples of its characteristic vectors α_i β_i and we try to discuss the geometrical and physical aspects of the model.

We consider the spatially-homogeneous isotropic Bianchi type-I cosmological model of Shahu et. al. [6], given by

$$ds^2 = -dT^2 + T^{8/9}\left(dX^2 + dY^2 + dZ^2\right), \tag{1}$$

where T is the cosmic time.

The pressure p, energy density ρ, the spatial volume V, the scalar expansion θ and the shear σ in the model (1) are given by

$$\rho = 2p = \frac{\beta_1}{T^2} \tag{2}$$

$$V = (-g)^{1/2} = ABC = T^{4/3} \tag{3}$$

$$\theta = \frac{4}{3}\frac{1}{T} \tag{4}$$

$$\sigma^2 = \frac{1}{2}\left[\left(\frac{g_{11,4}}{g_{11}} - \frac{g_{22,4}}{g_{22}}\right)^2 + \left(\frac{g_{22,4}}{g_{22}} - \frac{g_{33,4}}{g_{33}}\right)^2 + \left(\frac{g_{33,4}}{g_{33}} - \frac{g_{11,4}}{g_{11}}\right)^2\right] = 0,$$

respectively, where β_1 is constant.

The Kretschmann curvature invariant L is

$$L = \frac{656}{2187}\frac{\beta_1^4}{T^4}. \tag{5}$$

It indicates that the given model posses geometrical singularity at $T = 0$.

The physical aspects of model (1) are important and it is interesting to calculate the characteristic quantities for this metric (1). The characteristic quantities which are purely mathematical in nature can be put on physical basis. This is the motive of the present work to investigate the possibility of interpreting characteristic quantities physically by relating them with physical quantities. For completeness, we recall the definition of duplex and triple of orthogonal unit vectors α_i and β_i with the definition of SSST*** given in Borkar and Hajare [3].

*****Definition:** A four dimensional Riemannian space V_4 with signature -2 of the fundmental metric tensor g_{ij} is said to be spherically symmetric if the following conditions satisfies:

1. The curvature tensor K_{ijlm} of V_4 satisfies the equation

$$K_{ijlm} = -\rho^1\alpha_{[i}\alpha_{[l}\beta_{j]}\beta_{m]} - \rho^2 g_{[i[l}\alpha_{j]}\alpha_{m]} + \rho^3 g_{[i[l}\beta_{j]}\beta_{m]} + \rho^4 g_{[i[l}g_{j]m]} + \rho^5 g_{[i[l}Q_{j]m]}, \tag{6}$$

where α_i, β_i are mutually orthogonal unit vectors satisfying

$$\alpha_{j|i} = \sigma\alpha_i\beta_j + \kappa(g_{ij} + \alpha_i\alpha_j - \beta_i\beta_j) + \bar{\sigma}\beta_i\beta_j \tag{7}$$

$$\beta_{j|i} = \bar{\sigma}\beta_i\alpha_j + \bar{\kappa}(g_{ij} + \alpha_i\alpha_j - \beta_i\beta_j) + \sigma\alpha_i\alpha_j \tag{8}$$

$$\alpha_i\alpha^i = -1,\ \beta_i\beta^i = 1,\ \alpha_i\beta^i = 0 \tag{9}$$

$$Q_{ij} = \alpha_{(i}\beta_{j)}.$$

Here $\alpha_{j|i}$ is the usual Riemannian covariant derivative of α_j and the notation ( ) and [ ] stand for the usual symmetric and antisymmetric relations respectively. These equations determine the scalars $\rho^a, a = 1, \cdots, 5, \sigma, \bar{\sigma}, \kappa, \bar{\kappa}$.

2. One of the five scalars $\rho^1, \cdots, \rho^4$ and the scalars curvature $K = K_{ij}^{\cdot\cdot ji}$ is such that its gradient is a linear combination of α_i and β_i.

3. ρ^4, κ and $\bar{\kappa}$ satisfy
$$\rho^4 - 2(\kappa^2 - \bar{\kappa}^2) \neq 0.$$

In this definition the orthogonal unit vectors α_i and β_i the scalars $\rho^a, a = 1, \cdots, 5, \sigma, \bar{\sigma}, \kappa, \bar{\kappa}$ are called the analytical invariants of the spherically symmetric space-time.We can also say a set $\{\alpha_i, \beta_i, \rho^a, \sigma, \bar{\sigma}, \kappa, \bar{\kappa}\}$ of these quantities is called the characteristic system(CS) of spherically symmetric space-time.

***Definition:** A pair (α_p, β_q) of orthogonal unit vectors α_p, β_q is called **duplex** if only nonzero component of α_i is its p^{th} component and the only nonzero component of β_j is its q^{th} component. It is denoted by (α_p, β_q) ; $p, q = 1, 2, 3, 4$.

****Definition:** A set $\{\alpha_i, \beta_j\}$ of orthogonal unit vectors α_i, β_j in SSST is called **triple** if any two different components of α_i are nonzero with only one nonzero component of β_j or there is only one nonzero component of α_i with any two different nonzero components of β_j . It is denoted by $(\alpha_p, \alpha_q, \beta_r)$ or $(\alpha_r, \beta_p, \beta_q)$; $p, q, r = 1, 2, 3, 4, p \neq q$. The triples reducing to duplexes are:

$$(\alpha_1, \alpha_2, \beta_1), (\alpha_1, \alpha_2, \beta_2), (\alpha_1, \alpha_3, \beta_1), (\alpha_1, \alpha_3, \beta_3), (\alpha_1, \alpha_4, \beta_1), (\alpha_1, \alpha_4, \beta_4),$$
$$(\alpha_2, \alpha_3, \beta_2), (\alpha_2, \alpha_3, \beta_3), (\alpha_2, \alpha_4, \beta_2), (\alpha_2, \alpha_4, \beta_4), (\alpha_3, \alpha_4, \beta_3), (\alpha_3, \alpha_4, \beta_4),$$
$$(\alpha_1, \beta_1, \beta_2), (\alpha_2, \beta_1, \beta_2), (\alpha_1, \beta_1, \beta_3), (\alpha_3, \beta_1, \beta_3), (\alpha_1, \beta_1, \beta_4), (\alpha_4, \beta_1, \beta_4),$$
$$(\alpha_2, \beta_2, \beta_3), (\alpha_3, \beta_2, \beta_3), (\alpha_2, \beta_2, \beta_4), (\alpha_4, \beta_2, \beta_4), (\alpha_3, \beta_3, \beta_4), (\alpha_4, \beta_3, \beta_4).$$

The triples (not reducing to duplexes) are given below:

$$(\alpha_1, \alpha_2, \beta_3), (\alpha_1, \alpha_2, \beta_4), (\alpha_1, \alpha_3, \beta_2), (\alpha_1, \alpha_3, \beta_4), (\alpha_1, \alpha_4, \beta_2), (\alpha_1, \alpha_4, \beta_3),$$
$$(\alpha_2, \alpha_3, \beta_1), (\alpha_2, \alpha_3, \beta_4), (\alpha_2, \alpha_4, \beta_1), (\alpha_2, \alpha_4, \beta_3), (\alpha_3, \alpha_4, \beta_1), (\alpha_3, \alpha_4, \beta_2),$$
$$(\alpha_3, \beta_1, \beta_2), (\alpha_4, \beta_1, \beta_2), (\alpha_2, \beta_1, \beta_3), (\alpha_4, \beta_1, \beta_3), (\alpha_2, \beta_1, \beta_4), (\alpha_3, \beta_1, \beta_4),$$
$$(\alpha_1, \beta_2, \beta_3), (\alpha_4, \beta_2, \beta_3), (\alpha_1, \beta_2, \beta_4), (\alpha_3, \beta_2, \beta_4), (\alpha_1, \beta_3, \beta_4), (\alpha_2, \beta_3, \beta_4).$$

According to the values of ρ's, 24 non-reducing triples can be classified into two categories.

Category I:

$$(\alpha_1, \alpha_2, \beta_3), (\alpha_1, \alpha_2, \beta_4), (\alpha_1, \alpha_3, \beta_2), (\alpha_1, \alpha_3, \beta_4), (\alpha_2, \alpha_3, \beta_1), (\alpha_2, \alpha_3, \beta_4),$$

$$(\alpha_3,\beta_1,\beta_2),(\alpha_4,\beta_1,\beta_2),(\alpha_2,\beta_1,\beta_3),(\alpha_4,\beta_1,\beta_3),(\alpha_1,\beta_2,\beta_3),(\alpha_4,\beta_2,\beta_3)$$

Category II:

$$(\alpha_1,\alpha_4,\beta_2),(\alpha_1,\alpha_4,\beta_3),(\alpha_2,\alpha_4,\beta_1),(\alpha_2,\alpha_4,\beta_3),(\alpha_3,\alpha_4,\beta_1),(\alpha_3,\alpha_4,\beta_2),$$

$$(\alpha_2,\beta_1,\beta_4),(\alpha_3,\beta_1,\beta_4),(\alpha_1,\beta_2,\beta_4),(\alpha_3,\beta_2,\beta_4),(\alpha_1,\beta_3,\beta_4),(\alpha_2,\beta_3,\beta_4).$$

2. Mathematical and Physical Quantities

For the line element (1), the components of curvature tensor K_{ijlm} are

$$K_{1212}=K_{1313}=K_{2323}=\frac{16}{81}T^{\frac{-2}{9}},$$

$$K_{1414}=K_{2424}=K_{3434}=\frac{20}{81}T^{\frac{-10}{9}}. \tag{10}$$

For the line-element (1), for the triple $(\alpha_1,\alpha_2,\beta_3)$ of category I, the components of curvature tensorK_{ijlm} by using definition given in Borkar and Hajare [3] are:

$$\begin{aligned}
4K_{1212} =&T^{\frac{16}{9}}\left(\rho^2+2\rho^4\right),\\
4K_{1313} =&T^{\frac{16}{9}}\left[\frac{1}{2}\left(\rho^1+\rho^2\right)+\left(\rho^3+2\rho^4\right)\right],\\
4K_{1414} =&-T^{\frac{8}{9}}\left[\frac{1}{2}\rho^2+2\rho^4\right],\\
4K_{1213} =&\frac{T^{\frac{16}{9}}}{2}\sqrt{\frac{-1}{2}}\rho^5,\\
4K_{2123} =&\frac{T^{\frac{16}{9}}}{2}\sqrt{\frac{-1}{2}}\rho^5,\\
4K_{2323} =&T^{\frac{16}{9}}\left[\frac{1}{2}\left(\rho^1+\rho^2\right)+\left(\rho^3+2\rho^4\right)\right],\\
4K_{2424} =&-T^{\frac{8}{9}}\left[\frac{1}{2}\rho^2+2\rho^4\right],\\
4K_{3132} =&\frac{T^{\frac{16}{9}}}{2}(\rho^1+\rho^2),\\
4K_{3434} =&-T^{\frac{8}{9}}\left[\rho^3+2\rho^4\right],
\end{aligned} \tag{11}$$

$$4K_{4142} = \frac{-T^{\frac{8}{9}}}{2}\rho^2,$$
$$4K_{4143} = \frac{-T^{\frac{8}{9}}}{2}\sqrt{\frac{-1}{2}}\rho^5,$$
$$4K_{4243} = \frac{-T^{\frac{8}{9}}}{2}\sqrt{\frac{-1}{2}}\rho^5.$$

From (10) and (11), it is seen that

$$\rho^1 = \rho^2 = \rho^5 = 0, \quad \rho^3 = \frac{-16}{9}T^{-2}, \quad \rho^4 = \frac{32}{81}\frac{1}{T^2},$$

or

$$\rho^1 = \rho^2 = \rho^5 = 0, \quad \rho^3 = \frac{-9}{2}\rho^4 = \frac{-16}{9}\frac{1}{T^2}. \tag{12}$$

We got the same results as (12), for the remaining triples of category I. Thus we state the following

Theorem 1. *For the triples of category I, for the line element (1), we have*

$$\rho^1 = \rho^2 = \rho^5 = 0, \rho^3 = \frac{-9}{2}\rho^4 = \frac{-16}{9}\frac{1}{T^2}. \tag{13}$$

In the similar way, for the vectors of category II, we write:

Theorem 2. *For the triples of category II, for the line element (1), we have*

$$\rho^1 = \rho^2 = \rho^5 = 0, \rho^3 = \frac{-18}{5}\rho^4 = \frac{16}{9}\frac{1}{T^2}. \tag{14}$$

Let us write the results of pressure, density and other physical quantities, in view of category wise vectors.

For the vectors of category I: The pressure p and energy density ρ are:

$$\rho = 2p = \frac{81}{32}\beta_1\rho^4 = \beta_1\frac{1}{T^2}. \tag{15}$$

The spatial volume V is

$$V = \left(\frac{32}{81\rho^4}\right)^{2/3} = T^{4/3}. \tag{16}$$

The scalar expansion θ is

$$\theta = 3\sqrt{\frac{\rho^4}{2}} \propto \frac{1}{T}. \tag{17}$$

The shear σ is $\sigma^2 = 0$.

The Kretschmann curvature invariant L is

$$L = \frac{656}{2187}{\beta_1}^4 \left(\frac{-9}{16}\rho^3\right)^2 = \frac{656}{2187}{\beta_1}^4 \left(\frac{81}{32}\rho^4\right)^2 = \frac{656}{2187}{\beta_1}^4 \frac{1}{T^4}, \tag{18}$$

or $L \propto \frac{1}{T^4}$.

For the vectors of category II: The pressure p and energy density ρ are:

$$\rho = 2p = \frac{9}{16}\beta_1\rho^3 = \beta_1\frac{1}{T^2}. \tag{19}$$

The spatial volume V is

$$V = \left(\frac{16}{9\rho^3}\right)^{2/3} = T^{4/3}. \tag{20}$$

The scalar expansion θ is

$$\theta = \sqrt{\rho^3} \propto \frac{1}{T}. \tag{21}$$

The shear σ is $\sigma^2 = 0$.

The Kretschmann curvature invariant L is

$$L = \frac{656}{2187}{\beta_1}^4 \left(\frac{9}{16}\rho^3\right)^2 = \frac{656}{2187}{\beta_1}^4 \left(\frac{-81}{40}\rho^4\right)^2 = \frac{656}{2187}{\beta_1}^4 \frac{1}{T^4}, \tag{22}$$

i.e. $L \propto \frac{1}{T^4}$.

3. Geometrical and Physical Features

The whole universe is based on the single mathematical quantity ρ^3 (or ρ^4), which is inversely proportional to time factor T. The pressure p and the energy density ρ in the spatially homogeneous isotropic Bianchi type I cosmological model (1) can be put in terms of mathematical quantity ρ^3 (or ρ^4) and given by equations (15) and (19) respectively. Also the expansion θ and volume V of the model related with mathematical quantity ρ^3 (or ρ^4).

It is realized that the expansion in the model (1) decreases when the values of ρ^3 or ρ^4 decrease, i.e. when the time T increases and it becomes zero at very very large T i.e. ρ^3 or $\rho^4 \to 0$. The volume V attains the maximum value and it becomes infinite when ρ^3 or $\rho^4 \to 0$. There is no shear in the model. The pressure p and density ρ in the model increasing with the values of ρ^3 or ρ^4. When ρ^3 or $\rho^4 \to 0$ then $p = \rho = 0$ i.e., we get vacuum model, when ρ^3 or $\rho^4 \to 0$.

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References

[1] H. Takeno, *The Theory of Spherically Symmetric Space Times*, Research Institute of Theoretical Physics, Hiroshima University, Japan (1966).

[2] M.S. Borkar, *The Characteristic System of Spherically Symmetric Space Times and its Applications to Schwarzschild Singularity*, Ph.D. Thesis (2001).

[3] M.S. Borkar, M.H. Hajare, The analytical invariants of orthogonal spherically symmetric space-time, *J. Tensor Society Japan*, **68**, No. 3 (2007), 269-281.

[4] M.S. Borkar, M.H. Hajare, The study of spherically symmetric space-time through its characteristic vectors, *J. Vid J. Sci., Amravati*, **3**, No. 1 (2008), 40-46.

[5] M.S. Borkar, M.H. Hajare, The characteristic quantities of spherically symmetric space-time, *Intern. J. Appl. Math.*, **22**, No. 7 (2009), 1067-1073.

[6] R.C. Sahu, L.K. Mohapatra, G. Mohanty, Cosmological macro models in Einstein and Barber theories, *Jour. Indian Acad. Math.*, **3** (2009), 307-316.

[7] T.M. Karade and M.S. Borkar, On characteristic system of spherically symmetric space-times, *J. Post-RAAG Reports, Japan,* **350** (2000), 1-41.