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THE CONVEX ENVELOPE OF A FUNCTION BY PARABOLIC MONGE-AMPÈRE PROBLEM

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Abstract: The evolution of a hyper-surface moving according to its normals with a speed proportional at the Gauss curvature, leads to a nonlinear parabolic problem of Monge-Ampère type. In one dimension we use the motion of a convex graph to approximate the convex envelope of a giving function. The existence and uniqueness of the problem and the numerical result are considered.

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1. Introduction

The evolution of a hyper-surface, moving according to its normals with a speed proportional at the Gauss curvature on each point of the hyper-surface and in the direction of the outer normal vector, leads to the following nonlinear parabolic problem of Monge-Ampère type:

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$$(P_N)\quad \begin{cases} \dfrac{\partial u}{\partial t} - c\dfrac{det[D^2u]}{(1+|Du|^2)^{\frac{N+1}{2}}} = 0 & x \in \Omega, t \in [0,T] \\ u(x,0) = u_0(x) & x \in \Omega \\ u(x,t) = \Phi(x,t) & (x,t) \in \partial\Omega \times [0,T] \\ u & \text{convex,} \end{cases}$$

where Ω is a bounded open convex N-dimensional domain with smooth boundary $\partial\Omega$, u_0 and Φ are smooth given functions, and $c(x,t,u)$ is a bounded non negative function.

$$D_i u = \frac{\partial u}{\partial x_i},\ i = 1,...,N, \quad Du =^t (D_1u,..,D_Nu) \text{ and } D_{ij}u = \frac{\partial^2 u}{\partial x_i \partial x_j},$$

$i,j = 1,..,N$, with $[D^2u] = (D_{ij}u)$, $i,j = 1..,N$.

In this paper we consider the case $N = 1$. In the next section we start with $c(x,t,u) = 1$. We prove an existence and uniqueness theorem by using the method of maximal monotone operators. In Section 3 we describe an extrapolated Crank-Nicolson difference scheme for numerical approximation of the solution of (P_1). In the end, we give a numerical method for the approximation of f^{**} the convex envelope of f by taking an adequate function $c(x,t,u)$ in problem (P_1).

2. Existence Result

We solve (P_1) by using the method of maximal monotone operators [2]. So we rewrite it as an O.D.E.:

$$\begin{cases} \dfrac{du}{dt} + Au = 0 & t \in]0,T[\\ u(o) = u_0 \end{cases}$$

with:

$$Au = -\frac{D^2u}{1+Du^2} = -D(Arctg(Du)). \tag{1}$$

We show that the operator A is the subdifferential of a convex proper lower-semicontinuous function on $L^2(\Omega)$.

Let $\Psi : L^2(\Omega) \to [-\infty, +\infty]$ be defined by:

$$\Psi(u) = \begin{cases} \phi(u) & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where ϕ is defined on $H_0^1(\Omega)$ by:

$$\phi(u) = \int_0^1 F(Du(x))dx \quad \text{where} \quad F(x) = \int_0^x Arctg(s)ds. \tag{2}$$

Proposition 1. *The function Ψ is proper lower-semicontinuous and convex on $L^2(\Omega)$.*

Proof.

1. We have $F(x) \geq 0$, so Ψ is proper. Also $\Psi \neq +\infty$.

2. Ψ is convex by the convexity of F.

3. The inequality $F(x) \leq x^2 \quad \forall x$ implies that for $u \in H_0^1(\Omega)$ we have:

$$\phi(u) = \int_0^1 F(Du(x)dx \leq \int_0^1 (Du(x))^2 dx < +\infty. \tag{3}$$

4. To prove that Ψ is lower-semicontinuous on $L^2(\Omega)$ it suffices to show that for every $\lambda \in \mathbb{R}$ the set :

$$C = \{u \in L^2(\Omega)/\phi(u) \leq \lambda\} = \{u \in H_0^1(\Omega)/\phi(u) \leq \lambda\} \tag{4}$$

   is a closed subset of $L^2(\Omega)$: Let $\{u_n\} \in C$ be such that $u_n \to u$ in $L^2(\Omega)$ as $n \to \infty$. This implies that $Du_n \rightharpoonup Du$ weakly in $L^2(\Omega)$. By the Mazur lemma, there exists a sequence of convex combination:

$$v_n = \sum_{k=n}^{N} \lambda_k Du_k \to Du \quad \text{in} \quad L^2(\Omega) \quad \text{with} \quad \sum_{k=n}^{N} \lambda_k = 1. \tag{5}$$

   So this implies the existence of a subsequence $v_{n_k}(x) \to Du(x)$ a.e. on Ω.

5. Since F is convex and (5): $\displaystyle\int_0^1 F(v_k(x))dx \leq \sum_{i=n_k}^{N} \lambda_i \int_0^1 F(Du_i(x))dx \leq \lambda.$

   Then the Fatou lemma implies: $\displaystyle\int_0^1 F(Du(x))dx \leq lim \int_0^1 F(v_{n_k}(x))dx \leq \lambda$. Therefore $u \in C$ as claimed and Ψ is lower-semicontinuous.

The sub-differential $\partial\Psi$ is a maximal monotone operator on $L^2(\Omega)$ and $D(\partial\Psi)$ is a dense subset of $D(\Psi)$ (see V. Barbu [1], Ch. II).

For $u, v \in H_0^1(\Omega)$ we have:

$$\phi(v) - \phi(u) \geq (Au, v-u) = \int_0^1 Arctg(Du)D(v-u)dx.$$

Let $E = \{u \in H_0^1(\Omega)/Au \in L^2(\Omega)\}$, then the restriction of A at E coincides with the sub-differential $\partial\Psi$.

Thus the problem (P_1) is an evolution problem associated to a maximal monotone operator. We can now conclude this section by an existence and uniqueness result which follows from Theorem 2.1, Ch. IV of [1].

Proposition 2. *Let $u_0 \in L^2(\Omega)$. Then:*

1. *Problem (P_1) has a unique solution $u \in C([0,T], L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ and satisfying: $\sqrt{t}\dfrac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega))$.*
2. *If $u_0 \in H_0^1(\Omega)$, then $\dfrac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega))$ and $u \in L^\infty(0,T; H_0^1(\Omega))$.*

Proposition 3. *Suppose $u_0 \in C^\infty(\Omega)$ and convex, suppose that C is a bounded non negative function, then we have $\forall t \geq 0$, $x \to u(x,t)$ is convex, where u is a solution of (P_1) and $u_0 \leq u$.*

Proof. Since u_0 is convex, the Gauss curvature:

$$K_{u_0}(x) = \frac{D^2 u_0(x)}{\left(1 + (Du_0)^2(x)\right)^{3/2}} \geq 0 \, \forall x \in [0,1]. \tag{6}$$

(P_1) says that each point of the graph move with an instantaneous speed equal to $-CK_u.\nu$ then $\exists t_1 > 0$ such $\dfrac{\partial u}{\partial t} \geq 0 \, \forall t \leq t_1$ so the PDE in (P_1) implies that $D^2 u \geq 0$ and then u is convex. Since $x \to u(x,t_1)$ is convex and we conclude the proof by induction.

3. Numerical Scheme

In this section we are concerned with the construction of a difference scheme for the boundary value problem P_1. Let N be a positive integer and h defined by the relation $(N+1)h = 1$. For $0 < 2k < T$, we let J be the largest integer such that $kJ \leq T$.

We use a modification of the standard Crank-Nicolson difference scheme which is based on well-known predictor-corrector difference schemes for ordinary differential equations. Thus, let $w(x,t)$ satisfy:

$$(P_1) \quad \begin{cases} (1+Dw^2)\dfrac{\partial w}{\partial t} - D^2 w = 0, & x \in]0,1[, t \in]0,T[, \\ w(x,0) = \phi(x), & x \in]0,1[, \\ w(0,t) = w(1,t) = 0, & t \in [0,T]. \end{cases}$$

We define the standard difference operators D_+, D_- and D_0 in the usual way:

$$D_+u(x) = \frac{u(x+h) - u(x)}{h}, \quad D_-u(x) = \frac{u(x) - u(x-h)}{h},$$

$$D_0u(x) = \frac{u(x+h) - u(x-h)}{2h}.$$

We shall construct two sequences $u_0, u_1, ..., u_J$ and $u_{\frac{1}{2}}, u_{\frac{3}{2}}, ..., u_{J-\frac{1}{2}}$ by induction. First, we put $u_0 = \phi$ and : $u_{\frac{1}{2}} = \phi + \frac{k}{2G(D_0\phi)}[D_+D_-\phi]$, where $G(p) = 1+p^2$.

Now, assume that $u_0, u_1, ..., u_j$ and $u_{\frac{1}{2}}, u_{\frac{3}{2}}, ..., u_{j+\frac{1}{2}}$ have already been defined. Then u_{j+1} is defined to be the unique solution of the linearized Crank-Nicolson difference equation:

$$G(D_0u_{j+\frac{1}{2}})(u_{j+1} - u_j) = \frac{k}{2}D_+D_-(u_{j+1} + u_j) \tag{7}$$

and $u_{j+\frac{3}{2}}$ is defined directly by the linear extrapolation formula:

$$u_{j+\frac{3}{2}} = \frac{3}{2}u_{j+1} - \frac{1}{2}u_j. \tag{8}$$

We give some numerical examples for different giving u_0 using this scheme in the end of this paper.

4. Approximate Envelope

In this section we give a new method to approximate the convex envelope $f^{\star\star}$ of a function f (see [3], Section 1).

Let $f : [0,1] \to \Re$ be a function bounded below by a convex function u_0 and assume that $f(0) = f(1) = 0$.

Our approach is to evolves the graph of u_0 according to its normals with a speed proportional at the Gauss curvature and which vanish at all point x_i if $u(x_i, t_i) = f(x_i)$ for $t > t_i$.

For that we consider the problem P_2:

$$(P_2)\quad \begin{cases} \dfrac{\partial u}{\partial t} - C(x,t,u;f)\dfrac{D^2u}{(1+Du^2)} = 0 & x \in]0,1[, t \in]0,T[\\ u(x,0) = u_0(x) & x \in]0,1[\\ u(0,t) = u(1,t) = 0 & t \in [0,T] \end{cases}$$

where

$$C(x,t,u;f) = \begin{cases} 1, & \text{if } u(x,t) < f(x), \\ 0, & \text{if } u(x,t) = f(x). \end{cases}$$

Since C is a non negative function, for all $t \geq 0$, we have: $x \to u(x,t)$ is convex and $u(x,t) \leq f(x)$, where u is a solution of (P_2). For large T the function $u(x,T)$ is an approximation of the convex envelope of f.

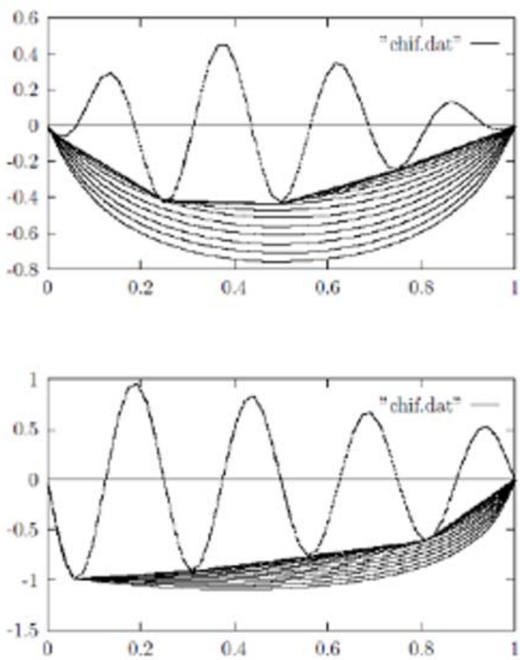

Figure 1: Envelope

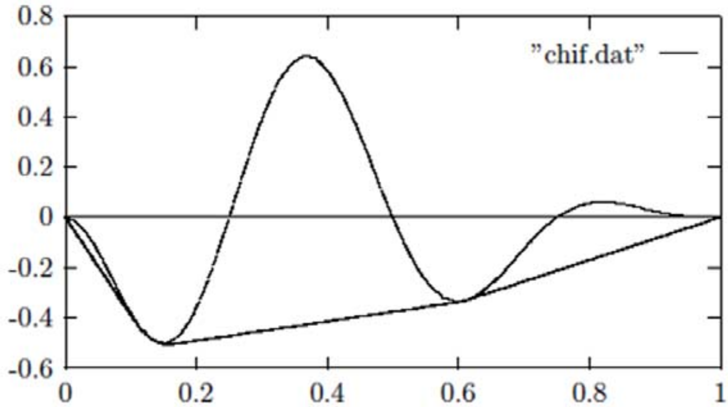

Figure 2: Approximate convex envelope

Numerical Examples

For a given f we define the convex function $u_0(x) = -\alpha.\sqrt{1-(2x-1)^2}$, where α is a constant such that: $u_0(x) \leq f(x)$, $\forall x \in [0,1]$.

First, in the following graphic we represent f and $u(.,T)$, representing $f^{\star\star}$ using the Crank-Nicolson scheme of Section 3.

For two given f the figures represent the $u(x_i,t_i)$ solution of (P_2) and $f(x_i)$ for $i = 1,...,10$. The approximate convex envelope $f^{\star\star}$ is $u(.,T)$ for large T:

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