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# REFLEXIVITY OF MULTIPLICATION OPERATORS ON WEIGHTED HARDY SPACES

Bahmann Yousefi[1 §], Ali Iloon Kashkooly[2]

[1]Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, IRAN
e-mail: b_yousefi@pnu.ac.ir

[2]Department of Mathematics
Yasouj University
Yasouj, IRAN
e-mail: kashkooli@mail.yu.ac.ir

**Abstract:** We give sufficient conditions under which the powers of the multiplication operator are reflexive.

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**Key Words:** Banach and Hardy spaces, Laurent series associated with a sequence $\beta$, weak operator topology, reflexive operator, multiplication operator

## 1. Introduction

In this section we include some preparatory material which is needed later. Let $X$ be a reflexive Banach space. For the algebra $\mathcal{B}(X)$ of all bounded operators on the Banach space $X$, the weak operator topology is the one in which a net $A_\alpha$ converges to $A$ if $A_\alpha x \to Ax$ weakly, $x \in X$. Recall that if $A \in \mathcal{B}(X)$, then $Lat(A)$ is by definition the lattice of all invariant subspaces of $A$, and $AlgLat(A)$ is the algebra of all operators $B$ in $\mathcal{B}(X)$ such that $Lat(A) \subset Lat(B)$. An operator $A$ in $\mathcal{B}(X)$ is said to be *reflexive* if $AlgLat(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $\mathcal{B}(X)$ that contains $A$ and the identity $I$ and is

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[§]Correspondence author

closed in the weak operator topology. For some sources of reflexivity see [1–6].

Let $\beta = \{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$. If $1 \leq p < \infty$, the space $L^p(\beta)$ consists of all formal Laurent series $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ such that the norm $\|f\|^p = \|f\|_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p$ is finite. When $n$ just runs over $\mathbb{N} \cup \{0\}$, the space $L^p(\beta)$ only contains formal power series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ and it is usually denoted by $H^p(\beta)$. If $p = 2$, such spaces were introduced by Allen L. Shields [1] to study weighted shift operators. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_{k\in\mathbb{Z}}$ is a basis for $L^p(\beta)$ such that $\|f_k\| = \beta(k)$. Now consider $M_z$, the operator of multiplication by $z$ on $L^p(\beta)$: $(M_z f)(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^{n+1}$ where $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \in L^p(\beta)$. In other words, $(M_z f)\hat{}(n) = \hat{f}(n-1)$ for all $n \in \mathbb{Z}$. Clearly $M_z$ shifts the basis $\{f_k\}_k$. The operator $M_z$ is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case $\|M_z^n\| = \sup_k[\beta(k+n)/\beta(k)]$ for all $n \in \mathbb{N}\cup\{0\}$. Clearly $M_z$ is invertible if and only if $\beta(k)/\beta(k+1)$ is bounded.

We denote the set of multipliers $\{\varphi \in L^p(\beta) : \varphi L^p(\beta) \subseteq L^p(\beta)\}$ by $L^p_\infty(\beta)$ and the linear operator of multiplication by $\varphi$ on $L^p(\beta)$ by $M_\varphi$. Also the set of multipliers on $H^p(\beta)$ is denoted by $H^p_\infty(\beta)$.

We say that a complex number $\lambda$ is a bounded point evaluation on $L^p(\beta)$ if the functional $e(\lambda) : L^p(\beta) \longrightarrow \mathbb{C}$ defined by $e(\lambda)(f) = f(\lambda)$ is bounded.

By the same method used in [2] we can see that $L^p(\beta)^* = L^q(\beta^{\frac{p}{q}})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Also if $f(z) = \sum_n \hat{f}(n)z^n \in L^p(\beta)$ and $g(z) = \sum_n \hat{g}(n)z^n \in L^q(\beta^{\frac{p}{q}})$, then clearly $< f, g >= \sum_n \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$. For a good source in formal power series, we refer the reader to papers [7–10].

## 2. Main Results

In this section we give sufficient conditions for reflexivity of the powers of the multiplication operator by the independent variable $z$, $M_z$, acting on Banach spaces of formal series.

The following theorem extends the results obtained by Shields (for the case $p = 2$) in [1] and due to similarity, we omit the proof. For each $\varphi \in L^p_\infty(\beta)$ put:

$$P_n(\varphi) = \sum_{k=0}^{n}(1 - \frac{k}{n+1})\hat{\varphi}(k)z^k, \; n \geq 0.$$

**Theorem 2.1.** *If $\varphi \in L^p_\infty(\beta) \cap H(\Omega_{11})$, then $M_{P_n(\varphi)} \longrightarrow M_\varphi$ in the weak operator topology.*

In the following theorem we use the notations:

$$\begin{array}{ll} r_{01} = \overline{\lim}\beta(-n)^{\frac{-1}{n}} & , \quad \Omega_{01} = \{z \in \mathbb{C} : |z| > r_{01}\} \\ r_{11} = \underline{\lim}\beta(n)^{\frac{1}{n}} & , \quad \Omega_{11} = \{z \in \mathbb{C} : |z| < r_{11}\} \\ \Omega_1 = \Omega_{01} \cap \Omega_{11}. & \end{array}$$

Note that if $r_{01} < r_{11}$, each point of $\Omega_1$ is a bounded point evaluation on $L^p(\beta)$.

**Theorem 2.2.** *Let $M_z$ be invertible on $L^p(\beta)$ and $r_{01} < r_{11}$. If there exists $c > 0$ such that $\|M_s\| \le c\|s\|_{\Omega_1}$ for all Laurent polynomials $s$, then $M_z$ is reflexive.*

*Proof.* Let $A \in AlgLat(M_{z^k})$. Since $Lat(M_z) \subset Lat(M_{z^k})$, thus $Lat(M_z) \subset Lat(A)$. This implies that $A \in AlgLat(M_z)$. By the same method used in the proof of Theorem 1 in [2] we can see that each point of $\Omega_1$ is a bounded point evaluation on $L^p(\beta)$. Since $M_z^* e(\lambda) = \lambda e(\lambda)$ for all $\lambda$ in $\Omega_1$, the one dimensional span of $e(\lambda)$ is invariant under $M_z^*$. Therefore it is invariant under $A^*$ and we write $A^* e(\lambda) = \varphi(\lambda) e(\lambda)$, $\lambda \in \Omega_1$. So

$$< Af, e(\lambda) >=< f, A^* e(\lambda) >= \varphi(\lambda) f(\lambda)$$

for all $f \in L^p(\beta)$ and $\lambda \in \Omega_1$. This implies that $A = M_\varphi$ and $\varphi \in L^p_\infty(\beta)$. Now since $\varphi \in L^p_\infty(\beta) \subset H^\infty(\Omega_1)$, by the same lemma in [1] we can write $\varphi(z) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n) z^n = \varphi_1(z) + \varphi_2(z)$ where

$$\begin{aligned} \varphi_1(z) &= \sum_{n=0}^{\infty} \hat{\varphi_1}(n) z^n \in H^\infty(\Omega_{11}), \\ \varphi_2(z) &= \sum_{n=-\infty}^{-1} \hat{\varphi_2}(n) z^n \in H^\infty(\Omega_{01}). \end{aligned}$$

Now, first we show that $\varphi_2 \equiv c$, a constant. To see this, note that $L^p(\beta) \in Lat(M_z)$, so $L^p(\beta) \in Lat(A)$ and also $L^p(\beta) \in Lat(M_{\varphi_1})$. Hence $\varphi_2 = \varphi - \varphi_1 = A1 - M_{\varphi_1} 1 \in L^p(\beta)$. If $\varphi_2 \ne c$, then $\hat{\varphi}_2(k) \ne 0$ for some $k < 0$. Since $\varphi_2 \in H^\infty(\Omega_{01})$, there is a sequence of polynomials in $\frac{1}{z}$, $\{s_n(\frac{1}{z})\}_n$, uniformly bounded on $\Omega_{01}$ and converging pointwise to $\varphi_2(z)$. Therefore $s_n(M_z^{-1}) \longrightarrow M_{\varphi_2}$ in the weak operator topology. To see this, note that

$$M^*_{s_n(\frac{1}{z})} e(\lambda) = s_n(\frac{1}{\lambda}) e(\lambda) \longrightarrow \varphi_2(\lambda) e(\lambda) = M^*_{\varphi_2} e(\lambda).$$

Hence $M^*_{s_n(\frac{1}{z})} f \longrightarrow M^*_{\varphi_2} f$ for every $f$ in the linear span of $\{e(\lambda) : \lambda \in \Omega_{01}\}$ that is dense in $L^q(\beta^{\frac{p}{q}})$. But $\varphi_2 \in L^p_\infty(\beta)$, thus by using the assumption we get

$$\|M_{s_n(\frac{1}{z})}\| \leq c\|s_n(\frac{1}{z})\|_{\Omega_{01}}.$$

Therefore $\{M_{s_n(\frac{1}{z})}\}_n$ is uniformly bounded and hence

$$M^*_{s_n(\frac{1}{z})} f \longrightarrow M^*_{\varphi_2} f$$

for every $f \in L^p(\beta)$. We have actually shown that

$$s_n(M_z^{-1}) \longrightarrow M_{\varphi_2}$$

in the strong operator topology. Therefore

$$< s_n(M_z^{-1})1, f_k > \longrightarrow < M_{\varphi_2} 1, f_k >$$

where $f_k(z) = z^k$. That is $s_n(\hat{\frac{1}{z}})(k) \longrightarrow \hat{\varphi}_2(k)$ as $n \longrightarrow \infty$. This is a contradiction, since the left hand side is zero and the right hand side is nonzero. Hence $\varphi_2$ is a constant, and so $M_\varphi = M_{\varphi_1} + cI$. Since $\varphi_1 \in L^p_\infty(\beta) \cap H(\Omega_{11})$, then by Theorem 2.1, $M_{P_n(\varphi_1)} \longrightarrow M_{\varphi_1}$ in the weak operator topology. Hence $M_{\varphi_1} \in W(M_z)$. This implies that $M_\varphi = M_{\varphi_1} + cI \in W(M_z)$ and so the proof is complete. $\square$

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