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REMARKS ON POSITIVE SOLUTIONS FOR AN m-POINT BOUNDARY VALUE PROBLEM

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Abstract: A nonlinear second order multi-point boundary value problem is considered. The existence of solution is proved through Alternative of Leray-Schauder's type. The existence of positive solutions is given by using Krasnoselskii's fixed point theorem. Iterative solutions are explored and a numerical method is presented.

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1. Introduction

In this paper we consider the following nonlinear boundary value problem

$$\begin{cases} u'' + q(t)f(t,u,u') = 0 \\ u(0) = 0, u(1) = g(u(\eta_1), u(\eta_2), ..., u(\eta_{m-2})) \end{cases} \tag{1}$$

where $g : \mathbb{R}^{m-2} \to \mathbb{R}$, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R}^{+} \to \mathbb{R}$ are continuous

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functions. This problem is commonly referenced in the literature as m-point (or multi-point) boundary value problem. The first results of the existence of solution were presented by Il'in and Moiseev [5, 6] where the equation considered was:

$$\begin{cases} u'' = f(t, u, u') \\ u(0) = 0, u(1) = \sum_{k=1}^{m-2} u(\eta_k). \end{cases} \tag{2}$$

According to Lin and Cui [9], equations with multi-point arise in problems that model viscoelastic and inelastic flows and deformation of beams. Due to the importance of this class of problem in several applications, many authors have developed studies considering variations and generalizations of (2). Most of these studies are related to the existence of solution and multi-solutions (see [3, 4, 12, 13, 11, 10, 14, 8] and the references therein). The techniques to obtain results of the existence are varied. For example, Gupta [3, 4], obtained the conditions of the existence using degree-theoretic arguments. Ma [12], using Leray-Schauder's Alternative, demonstrated the existence by considering the following m-point boundary value problem

$$\begin{cases} u''(t) = f(t, u(t), u'(t)) + e(t) \\ u'(0) = 0, u(1) = \sum_{k=1}^{m-2} a_k u(\eta_k) \end{cases}, \tag{3}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ and $e : [0,1] \to \mathbb{R}$ are continuous functions and $a_i \in \mathbb{R}$, with all of the a_i 's having the same sign. Recently, Sun et al [14] obtained positive solution by using the Krasnoselskii's Theorem to the following problem

$$\begin{cases} u''(t) + a(t)f(u) = 0 \\ u'(0) = 0, u(1) - \sum_{i=1}^{m-2} k_i u(\eta_i) = b \end{cases}, \tag{4}$$

under conditions

- $k_i > 0$ $(i = 1, ..., m-2)$, $0 < \eta_1 < \eta_2 < ... < \eta_{m-2} < 1$, $\sum_{i=1}^{m-2} k_i < 1$ and $\sum_{i=1}^{m-2} k_i \eta_i < 1$;
- a and f are continuous function with $a(t) \neq 0$ for some $t \in [0,1]$.

One of the advantages of using Krasnoselskii's theorem is that we can explore qualitative aspects of the solution as concavity and positivity. In [1] we have an interesting exposition of the use of this theorem.

Considering the theoretical purposes, the basic idea that we will use in this paper is the following: using the Leray-Schauder Alternative, we show a result of existence of solution to (1). Then, imposing some additional assumptions, we apply Krasnoselskii's Theorem and obtain results of existence of positive solution. Although the use of Krasnoselskii's Theorem represents a classical technique to solve second order equations, we are unaware of studies that apply this technique to the m-point boundary value problem in the form of (1). The goal of our results is to show how we can obtain positive solution for (1) by imposing sometimes more restrictions on f, other times by imposing more restrictions on g.

Although there are several papers devoted to the existence of solutions, on the other hand, little attention is given to numerical methods when the proposed problem has a more comprehended form, as the case of (1). In [2] numerical solutions for (1) with $m = 3$ were given by means of two algorithms: in the first, an iterative method based on fixed point with trapezoidal rule is presented; in the second, the shooting method is the essence that defines the algorithm. Still, in [2] iterative solutions are explored by using the Contraction Principle. To complement this study we will extend the results of [2] for $m \geq 3$ and present some modifications in the first algorithm of aforementioned paper as a way to achieve better numerical results. Furthermore, we propose non-classical examples to validate the theorems and to test the modified algorithm.

This paper is organized as follows: a basic result of existence by using Alternative of Leray-Schauder's type is presented in Section 2. The main results are given in Section 3 where two theorems are proved by Krasnoselskii's theorem. Section 4 contains an existence result for iterative solutions under some local assumptions on f and g. Still, in Section 4 we present an algorithm and numerical examples.

For the purpose of this work we need to introduce the main tools.

Theorem 1. *Let E be a Banach space, $C \subset E$ a closed and convex set, Ω an open set in C and $p \in \Omega$. Then each completely continuous mapping $T : \overline{\Omega} \to C$ has at least one of the following properties:*

$(A1)$ *T has a fixed point in $\overline{\Omega}$.*

$(A2)$ *There are $u \in \partial\Omega$ and $\lambda \in (0, 1)$ such that $u = \lambda T(u) + (1 - \lambda)p$.*

Theorem 2. *Let E be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that, either*

$(B1)$ $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1$ *, and* $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$*, or*

$(B2)$ $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1$ *, and* $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The first theorem is the well-known Leray-Schauder's Alternative and the second theorem is due Krasnoselskii, see [1, 7].

2. Existence of Solution

Let $E = C^1[0,1]$ be the Banach space of the continuously differentiable functions in $[0,1]$ with the norm

$$\|u\|_E = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

We begin this section by observing that the solutions of (1) can be written as

$$u(x) = \int_0^1 G(x,t)q(t)f(t,u(t),u'(t))dt + g(u(\eta_1),\dots,u(\eta_{m-2}))x,$$

where G is the Green's function

$$G(x,t) = \begin{cases} t(1-x), & t \le x \\ x(1-t), & x \le t \end{cases}.$$

Thus u is a solution of (1) if, and only if, is a fixed point of operator $T : E \to E$ defined by

$$(Tu)(x) = \int_0^1 G(x,t)q(t)f(t,u(t),u'(t))dt + g(u(\eta_1),\dots,u(\eta_{m-2}))x. \tag{5}$$

We will detail some properties of G. Note that

$$\partial_x G(x,t) = \begin{cases} -t, & t \le x \\ 1-t, & x \le t \end{cases},$$

then G satisfies:

$$G(x,t) = |G(x,t)| \le |\partial_x G(x,t)|. \tag{6}$$

Moreover, let $\overline{m}$ be a constant in $[0,1/2]$. Thus, we have

$$G(x,t) \ge \overline{m}G(t,t),\ \forall x \in [\overline{m}, 1-\overline{m}]. \tag{7}$$

Using (6) and (7) we can verify some properties of the operator defined in (5). In fact, for all $x \in [0,1]$ we have

$$\begin{aligned} |(Tu)(x)| &= \left| \int_0^1 G(x,t)q(t)f(t,u(t),u'(t))dt + g(u(\eta_1),\dots,u(\eta_{m-2}))x \right|, \\ &\le \int_0^1 G(x,t)|q(t)f(t,u(t),u'(t))|dt + \\ &\qquad |g(u(\eta_1),\dots,u(\eta_{m-2}))||x|, \\ &\le \int_0^1 |\partial_x G(x,t)||q(t)f(t,u(t),u'(t))|dt + \\ &\qquad |g(u(\eta_1),\dots,u(\eta_{m-2}))|. \end{aligned}$$

Therefore,

$$|(Tu)(x)| \leq \int_0^1 |\partial_x G(x,t)||q(t)f(t,u(t),u'(t))|dt + |g(u(\eta_1),\dots,u(\eta_{m-2}))|, \text{ for } x \in [0,1]. \tag{8}$$

Similarly we can obtain the inequality:

$$|(Tu)'(x)| \leq \int_0^1 |\partial_x G(x,t)||q(t)f(t,u(t),u'(t))|dt + |g(u(\eta_1),\dots,u(\eta_{m-2}))|. \tag{9}$$

In order to assure the existence of solution of (1) we will need the following hypotheses:

$(H1)$ There are positive constants $\alpha,\ A,\ B$ such that:

- $\max\limits_{(t,u,v)\in[0,1]\times[-\alpha,\alpha]\times[-\alpha,\alpha]} \{|f(t,u,v)|\} < \dfrac{\alpha A}{d_1}$, where $d_1 = \max\limits_{x\in[0,1]} \left\{ \int_0^1 |\partial_x G(x,t)q(t)|dt \right\}$;
- $|g(y)| \leq \alpha B,\ \forall\ y \in [0,\alpha]^{m-2}$;
- $A + B \leq 1$.

Theorem 3. *Suppose that $(H1)$ holds. Then (1) has solution $u^* \in E$ such that $\|u^*\|_E \leq \alpha$.*

Proof. We will use Theorem 1 with $p = 0$ and $\Omega = \{u \in E; \|u\|_E < \alpha\}$. Note that the operator $T : \overline{\Omega} \to E$ is completely continuous by Arzela - Ascoli's theorem. Now, suppose there are $u \in \partial\Omega$ and $\lambda \in (0,1)$ with $u(x) = \lambda Tu(x)$. From $(H1)$, (8) and (9) we can get

$$\begin{aligned}
\max\{|u(x)|,|u'(x)|\} &< \max\{|(Tu)(x)|,|(Tu)'(x)|\} \\
&\leq \int_0^1 |\partial_x G(x,t)q(t)f(t,u(t),u'(t))|dt + \\
&\qquad |g(u(\eta_1),\dots,u(\eta_{m-2}))| \\
&\leq \frac{\alpha A}{d_1}\int_0^1 |\partial_x G(x,t)q(t)|dt + \\
&\qquad |g(u(\eta_1),\dots,u(\eta_{m-2}))| \\
&\leq \alpha A + \alpha B \\
&\leq \alpha.
\end{aligned}$$

Since that $\|u\|_E < \alpha$, the conclusion $(A2)$ of Theorem 1 cannot occur. Thus we conclude that $(A1)$ must hold. Therefore there is $u^* \in E$ such that $\|u^*\|_E \leq \alpha$. $\square$

3. Existence of Positive Solution

In this section we establish the main result to the existence of positive solution to (1). For this purpose we need the following additional assumptions:

$(H2)$ $q(t)f(t,u,v) \geq 0,\ \forall (t,u,v) \in [0,1] \times [0,\alpha] \times [-\alpha,\alpha]$;

$(H3)$ $g(y) \geq 0,\ \forall y \in [0,\alpha]^{m-2}$;

$(H4)$ There is constant β, such that $0 < \beta < \alpha$ and

$$\min_{[\overline{m}-1,\overline{m}]\times[0,\beta]\times[-\beta,\beta]} f(t,u,v) \geq \frac{\beta}{\overline{m}d_2},$$

where $\overline{m} \in [0, \frac{1}{2}]$ and $d_2 = \displaystyle\int_{\overline{m}}^{1-\overline{m}} q(t)G(t,t)dt.$

Theorem 4. *Suppose that $(H1)-(H4)$ hold. Then (1) has a solution $u^* \in E$, such that $\beta \leq \|u^*\|_E \leq \alpha$.*

Proof. We start the proof defining the cone $K \subset E$ by

$$K = \{u \in E; u \geq 0, u(0) = 0\}.$$

From $(H2)$ and the definition of G, we have that T applies K in K. As seen in the last result, T is completely continuous.

We shall apply Theorem 2. Thus, we will define $\Omega_1 = \{u \in E; \|u\|_E < \beta\}$, $\Omega_2 = \{u \in E; \|u\|_E < \alpha\}$ and we will show that the following conditions are true for all $u \in K$:

(a) if $\|u\|_E = \alpha$ then $\|Tu\|_E \leq \alpha$;

(b) if $\|u\|_E = \beta$ then $\|Tu\|_E \geq \beta$.

In fact, the demonstration of the condition (a) is similar to the proof of the last theorem. To prove the second condition we consider $\|u\|_E = \beta$, then

$$\|Tu\|_E \geq \|Tu\|_\infty,$$

that is,

$$\|Tu\|_E \geq \max_{x\in[0,1]} \left\{ \int_0^1 G(x,t)q(t)f(t,u(t),u'(t))dt + \right.$$

$$\left. g(u(\eta_1),\ldots,u(\eta_{m-2}))x \right\}.$$

Now, from $(H3)$, (5) and $(H4)$ we have

$$\begin{aligned}
\|Tu\|_E &\geq \max_{x\in[0,1]} \left\{ \int_0^1 G(x,t)q(t)f(t,u(t),u'(t))dt \right\}, \\
&\geq \overline{m} \int_{\overline{m}}^{1-\overline{m}} G(t,t)q(t)f(t,u(t),u'(t))dt, \\
&\geq \frac{\beta}{\overline{m}d_2}\overline{m} \int_{\overline{m}}^{1-\overline{m}} G(t,t)q(t)dt = \beta.
\end{aligned}$$

Therefore, we have by Theorem 2 that T has fixed-point $u^* \in K$ with $\beta \leq \|u^*\|_E \leq \alpha$. □

Now we can observe that the hypotheses $(H3)$ and $(H4)$ can be replaced by

$(H3')$ There is constant r such that

$$g(0,\ldots,0) \geq r > 0, \text{ and } g(y) \geq 0, \forall \; y \in [0,\alpha]^{m-2}.$$

With this hypothesis we obtain the following result.

Theorem 5. *Suppose that $(H1)$, $(H2)$ and $(H3')$ hold. Then (1) has at least one positive solution $u^* \in E$.*

Proof. To prove this result it is necessary to verify that there is $\overline{\beta} > 0$ such that

$$\|Tu\|_E \geq \|u\|_E, \quad \forall u \in K \cap \partial\Omega_3,$$

where $\Omega_3 = \{u \in E; \|u\|_E < \overline{\beta}\}$.

Let us assume that the inequality is false, that is, for every $\overline{\beta}$ such that $\alpha > \overline{\beta} > 0$ there exists $u \in E$ with $\|u\|_E = \overline{\beta}$ and $\|Tu\|_E < \overline{\beta}$. Thus for all $n \in \mathbb{N}^*$ with $\frac{1}{n} < \alpha$, we can find $u_n \in K$ such that

$$\|u_n\|_E = \frac{1}{n} \text{ and } \|Tu_n\|_E < \frac{1}{n}.$$

Then $\|u_n\| \longrightarrow 0$ and $\|Tu_n\| \longrightarrow 0$, when $n \longrightarrow \infty$. Being T continuous, we have $\|T0\|_E = 0$. On the other hand, using $(H2)$ and $(H3')$ we have

$$\begin{aligned}
\|T0\|_\infty &= \max_{x\in[0,1]} \left\{ \int_0^1 G(x,t)q(t)f(t,0,0)dt + g(0,\dots,0)x \right\}, \\
&\geq g(0,\dots,0)), \\
&\geq r > 0,
\end{aligned}$$

which is a contradiction. Thus we have the result. □

Remark. Note that the most important step in the proof of Theorem 5 is to impose conditions to conclude that 0 is not fixed point of T. Therefore, we can purpose other conditons in order to establish similar results. More specifically, we can observe that the hypothesis $(H4)$ could be replaced by

$$\exists \ \ t \in [0,1] \ \text{ such that } \ q(t)f(t,0,0) = s > 0,$$

and the demonstration would be similar to the proof of Theorem 5.

Example 6.

$$\begin{aligned}
f(t,u,v) &= 2t + \frac{3}{10}u + \frac{1}{20}v^2 \\
q(t) &= e^t \\
g(y_1,y_2,y_3) &= \frac{1}{12}(y_1+y_2+y_3).
\end{aligned}$$

Choosing the constants

$$\alpha = 10, \quad A = d_1 = exp(1) - 2, \quad B = \frac{1}{4}, \quad m = \frac{1}{4}, \quad \beta = \frac{1}{50},$$

we can easily verify that in these conditions the hypotheses $(H1) - (H4)$ are satisfied.

Example 7.

$$\begin{aligned}
f(t,u,v) &= \frac{5}{3}t + \frac{1}{20}u|v| + \frac{1}{3}|v|\sin^2(u) \\
q(t) &= e^t \\
g(y_1,y_2,y_3) &= \sin(y_1y_2y_3) + \frac{1}{20}y_2 + \frac{1}{10}.
\end{aligned}$$

In this example we can choose

$$\alpha = 10, \quad A = d_1 = exp(1) - 2, \quad B = \frac{1}{4},$$

and consequently, the hypotheses $(H1), (H2)$ and $(H3)'$ of Theorem 5 are verified.

4. Iterative Solutions

In this section we present a result to show the existence of iterative solution by the Contraction Principle. Although classic, this study is very important in order to establish algorithms to solve the proposed problem. We begin this section by considering the iterative sequence

$$u^{k+1} = T(u^k), \tag{10}$$

$$u^{k+1}(x) = \int_0^1 G(x,t)q(t)f(u^k(t), u'^k(t))dt \tag{11}$$

$$+g(u^k(\eta_1), \ldots, u_k(\eta_{m-2}))x. \tag{12}$$

We will show the existence of limit to the sequence above through Banach's fixed-point theorem (we recommend [1]). Thus, we will need the following hypotheses.

$(S1)$ There exist positive constant α, A, e B such that

$$\max_{(t,u,v)\in[0,1]\times[-\alpha,\alpha]\times[-\alpha,\alpha]} |f(t,u,v)| \le \frac{\alpha}{d_1},$$

where

$$d_1 = \max_{x\in[0,1]} \int_0^1 |\partial_x G(x,t)q(t)|dt,$$

$$|g(y)| \le B, \quad \forall y \in [0,\alpha]^{m-2},$$

and

$$A + B \le 1.$$

$(S2)$ There exist $\lambda_f > 0$ e $\lambda_g > 0$ such that

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le \lambda_f \max\{|u_1 - u_2|, |v_1 - v_2|\},$$

$$|g(y_1) - g(y_2)| \le \lambda_g |y_1 - y_2|,$$

for all $t \in [0,1]$, $u_1, v_1, u_2, v_2 \in [-\alpha,\alpha]$ and $y_1, y_2 \in [-\alpha,\alpha]$.

$(S3)$ $d_1\lambda_f + \lambda_g < 1$.

Theorem 8. *Suppose that $(S1)-(S3)$ hold. Then (1) has an unique solution u such that, $\|u\|_E \le \alpha$. Besides, this solution is the uniform limit of the iterative sequence $u^{k+1} = T(u^k)$.*

Proof. First, we will show that T applies Ω in Ω, where $\Omega = \{u \in E; \|u\|_E \le \alpha\}$. Note that if $u \in E$, then Tu satisfies $(Tu)(0) = 0$. Thus we have that $\|(Tu)'\|_\infty = \|Tu\|_E$. Using $(S1)$ we obtain

$$\begin{aligned}\|Tu\|_E &= \|(Tu)'\|_\infty \\ &\le \frac{\alpha}{d_1}\int_0^1 |\partial_x G(x,t)q(t)|dt + B \\ &\le \alpha.\end{aligned}$$

Therefore T applies Ω in Ω. Now we will show that T is a contraction.

$$\begin{aligned}\|Tu - Tv\|_E &= \|(Tu-Tv)'\|_\infty \\ &= \max_{x\in[0,1]}\{|\int_0^1 \partial_x G(x,t)q(t)[f(t,u(t),u'(t)) - f(t,v(t),v'(t))]dt \\ &\quad + [g(u(\eta_1),\dots,u(\eta_{m-2})) - g(v(\eta_1),\dots,v(\eta_{m-2}))]|\}.\end{aligned}$$

Thus, we can apply $(S2)$ and consequently we have,

$$\begin{aligned}\|Tu - Tv\|_E &\le \max_t\{d_1|f(t,u(t),u'(t)) - f(t,v(t),v'(t))| + \\ &\qquad |g(u(\eta_1),\dots,u(\eta_{m-2})) - g(v(\eta_1),\dots,v(\eta_{m-2}))|\} \\ &\le d_1\lambda_f \max_t\{|u(t)-v(t)|, |u'(t)-v'(t)|\} + \\ &\qquad \lambda_g \max_t\{|u(t)-v(t)|\} \\ &\le (d_1\lambda_f + \lambda_g)\|u-v\|_E.\end{aligned}$$

Finally, from $(S3)$ we conclude that T is a contraction. □

The last theorem allows us to establish conditions for local convergence of algorithms that use the iterative sequence:

$$u^{k+1} = T(u^k).$$

In this sense, in [2], the following method was presented considering $m = 3$ and the formulae

$$\begin{aligned}u^{k+1}(x) = \int_0^x & t(x-1)f(t,u^k(t),u^{k'}(t))dt \\ &+ \int_x^1 x(t-1)f(t,u^k(t),u^{k'}(t))dt + g(u(\eta))x.\end{aligned}$$

Algorithm 1.

1. Define a uniformly spaced mesh $\{x_j\}$.
2. Choose initial approximation $u_j^0 = u^0(x_j)$.
3. For $k = 1, 2, 3, \ldots$
   (a) Compute $u^k(\eta)$, $\eta = (\eta_1, ..., \eta_{m-2})$ by using cubic-spline interpolation.
   (b) Compute $u_j'^k$ with central-diferences.
   (c) Compute $u_j'^{k+1}$ by to use
   $$u^{k+1} = T(u^k)$$
   where the integrals are computed through trapezoidal rule.
4. Test convergence.

Now, we can establish some modifications in Algorithm 1 and, consequently, generate a new algorithm. However, for the new algorithm, we need to be aware that some properties should be preserved. These are:

- Properties about local convergence;
- The new algorithm must be compatible in the computational point of view with Algorithm 1.

Algorithm 2

1. Define a uniformly spaced mesh $\{x_j\}$, $j = 1, n$.
2. Choose initial approximation $u_j^0 = u^0(x_j)$.
3. For $k = 1, 2, 3, \ldots$
   (a) Compute $u^k(\eta)$, $\eta = (\eta_1, ..., \eta_{m-2})$ by using cubic-spline interpolation.
   (b) Compute $u_j'^k$ with central-diferences.
   (c) For $i = 1, n$

- Compute u_i^{k+1} using
$$u^{k+1} = T(u^k)$$
where the integrals are computed through Simpson's rule.
- Update $u_i'^k$ using u_i^{k+1}.

4. Test convergence.

In sequence we are presenting some examples in order to establish the effectiveness of Algorithm 2. To analyze our results, we extend Algorithm 1 for the case $m \geq 3$. In tables, ε_u^k denotes $\|u^* - u^k\|_\infty$ where u^* is the exact solution, ε^k denotes $\|u^{k+1} - u^k\|_\infty$ and $\overline{\varepsilon}^k = \frac{\|u^{k+1}-u^k\|_\infty}{\|u^{k+1}\|_\infty}$. Still, "It" denotes "iteration". In all examples, the analytical solution is $u^*(x) = x$.

Example 9. Our first example, we consider

$$\begin{aligned} f(x,u,u') &= (u')^2 - u^2 + 2xu' + x^2 - 2x + 1 \\ g(y) &= 4y \quad \eta = \frac{1}{4}, \quad q(t) = -1. \end{aligned}$$

Table 1 contains results of application in algorithms mentioned before.

It	Algorithm 1			Algorithm 2		
	ε_u^k	ε^k	$\overline{\varepsilon}^k$	ε_u^k	ε^k	$\overline{\varepsilon}^k$
5	0.2328	0.1618	0.4336	0.1763	0.1319	0.4191
10	0.0781	0.1092	0.1320	2.822e-03	0.0122	0.0119
20	5.379e-03	0.0177	7.872e-03	5.200e-06	5.932e-06	2.108e-05
30	1.951e-03	1.293e-03	2.958e-3	2.898e-09	1.048e-08	1.634e-08

Table 1: Comparison between Algorithms 1 and 2 considering Example 9.

Example 10. In this example we consider

$$\begin{aligned} f(x,u,u') &= (u')^2 - u^2 + 2xu' + x^2 - 2x + 1 \\ g(y) &= 2y \quad \eta = \frac{1}{2}, \quad q(t) = -1. \end{aligned}$$

The numerical results are given in Table 2.

It	Algorithm 1			Algorithm 2		
	ε_u^k	ε^k	$\overline{\varepsilon}^k$	ε_u^k	ε^k	$\overline{\varepsilon}^k$
5	0.0884	0.2460	0.4729	0.0885	0.2511	0.4838
10	0.0388	0.0295	0.0429	8.020e-03	6.409e-03	0.0263
20	8.336e-04	2.704e-03	1.496e-03	1.160e-05	3.277e-05	3.356e-05
30	7.290e-05	1.825e-04	1.425e-04	1.699e-08	6.568e-08	5.576e-08

Table 2: Comparison between Algorithms 1 and 2 considering Example 10.

Example 11. In this example we consider the "m-point" closest to $x = 1$ because, in this case, the boundary condition is not accurate. The numerical results are exposed in Table 3.

$$\begin{aligned} f(x, u, u') &= (u')^2 - u^2 + 2xu' + x^2 - 2x + 1 \\ g(y) &= \frac{4}{3}y \quad \eta = \frac{3}{4}, \quad q(t) = -1. \end{aligned}$$

It	Algorithm 1			Algorithm 2		
	ε_u^k	ε^k	$\overline{\varepsilon}^k$	ε_u^k	ε^k	$\overline{\varepsilon}^k$
5	0.3398	0.2008	0.4695	0.3008	0.2212	0.4683
10	0.0146	4.261e-3	0.0249	2.436e-3	6.6363-03	5.187e-03
20	7.184e-05	4.351e-05	3.384e-05	1.690e-07	1.199e-06	1.303e-06
30	1.951e-07	3.610e-07	3.325e-07	2.640e-11	1.797e-10	3.441e-10

Table 3: Comparison between Algorithms 1 and 2 considering Example 11.

Example 12. In this example we choose six points on boundary and the results are given in Table 4.

$$\begin{aligned} f(x, u, u') &= (u')^2 - u^2 + 2xu' + x^2 - 2x + 1 \\ g(y) &= y_1 + y_2 + y_3 + (y_4)^2, \ \eta = \left(\frac{1}{6}, \ \frac{1}{4}, \ \frac{1}{3}, \ \frac{1}{2}\right), \ q(t) = -1. \end{aligned}$$

Now, we can make additional tests. From Theorem 4 we have a solution to Example 6 and from Theorem 5 we have a solution to Example 7 but in both

It	Algorithm 1			Algorithm 2		
	ε_u^k	ε^k	$\overline{\varepsilon}^k$	ε_u^k	ε^k	$\overline{\varepsilon}^k$
5	0.1923	0.2710	0.4666	0.1666	0.2511	0.4640
10	0.0595	0.1010	0.0454	0.0144	0.0105	0.0414
20	8.733e-03	0.0247	0.0130	2.679e-05	1.014e-04	7.445e-05
30	3.202e-03	4.205e-03	4.766e-03	2.549e-07	1.970e-07	7.868e-07

Table 4: Comparison between Algorithms 1 and 2 considering Example 12.

cases, we do not know which they are. Let us apply Algorithms 1 and 2 in these problems. For this purpose, we can consider the condition

$$\frac{\|u^{k+1} - u^k\|_\infty}{\|u^{k+1}\|_\infty} < 10^{-4}$$

as stopping criterion for the algorithms.

Testing Example 6 with $\eta = (0.1\ 0.5\ 0.8)$ and $n = 10$, Algorithms 1 and 2 stop, according to the criteria, after 7 and 6 iterations, respectively. In Table 5 we present the numerical results and in Figure 1 we have a graph of solution obtained by Algorithm 2.

It	Algorithm 1		Algorithm 2	
	ε^k	$\overline{\varepsilon}^k$	ε^k	$\overline{\varepsilon}^k$
1	0.24397493086	1	0.24847054101	1
2	0.04028199623	0.14475388765	0.04111557542	0.14698155860
3	0.00646343252	0.02276673954	0.00589899231	0.02069427925
4	0.00106860074	0.00375174073	0.00098009756	0.00342734980
5	0.00017694198	0.00062088824	0.00016654174	0.00058207157
6	0.00002931072	0.00010284193	0.00002841879	0.00009931585
7	0.00000485568	0.00001703676	-	-

Table 5: Comparison between Algorithms 1 and 2 considering Example 6.

Finally, testing Example 7 with $\eta = (\frac{1}{3}\ \frac{1}{2}\ \frac{3}{4})$ and $n = 10$, we have that both methods stop after 6 iterations. The numerical results are given in Table 6 and Figure 2.

It is important to observe that the solutions found by the algorithms considering Examples 6 and 7 were the same. Moreover, we can see by the graphs that these solutions are concave, as expected in view of Theorems 4 and 5.

From the numerical point of view, Algorithm 2 shows a little better performance than Algorithm 1, considering accuracy of results, but this performance

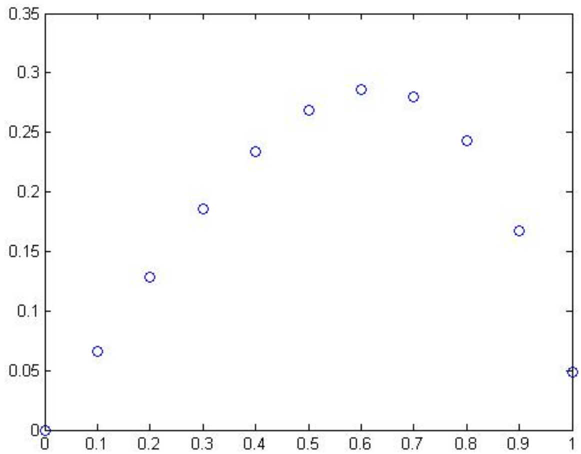

Figure 1: Numerical solution obtained from Example 6 using Algorithm 2.

It	Algorithm 1		Algorithm 2	
	ε^k	$\overline{\varepsilon}^k$	ε^k	$\overline{\varepsilon}^k$
1	0.26111771215	1	0.26476099521	1
2	0.02269595097	0.08152151785	0.02307431167	0.08262193551
3	0.00264718559	0.00944039838	0.00245886137	0.00874183918
4	0.00032768622	0.00116756571	0.00031144909	0.00110626632
5	0.00004064385	0.00014480066	0.00004032011	0.00014320003
6	0.00000504244	0.00001796430	0.00000504244	0.00001854449

Table 6: Comparison between Algorithms 1 and 2 considering Example 7.

can be limited in massive tests. However, Algorithm 2 is promising in the following sense: the updating of derivative allows us to accelerate the calculation of the derivatives at each iteration. In addition, the Simpson's rule can be a better result than trapezoidal rule. Thus Algorithm 2 tends to consume fewer iterations than the Algorithm 1 when establishing a limiting factor for the expression:

$$\frac{\|u^{k+1}-u^k\|_\infty}{\|u^{k+1}\|_\infty}.$$

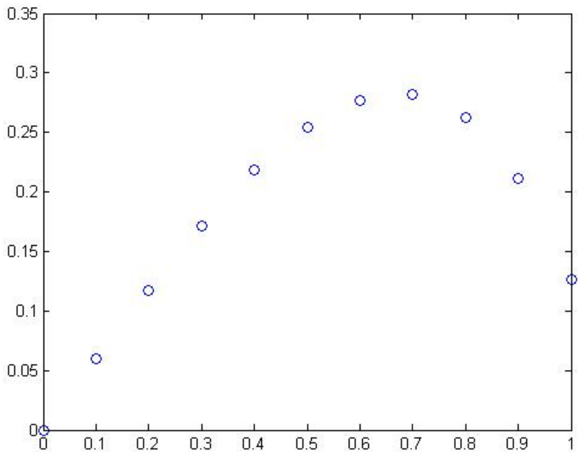

Figure 2: Numerical solution obtained from Example 7 using Algorithm 2.

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